1. (10 pts)

(a) Let A be an $n \times n$ real matrix and $||\cdot||_1$ denote the norm on $\mathbb{R}^n$ given by $||x||_1 = \sum_{i=1}^n |x_i|$ for any vector $x \in \mathbb{R}^n$. Prove that the matrix operator norm

$$||A||_1 = \max_{x \neq 0, x \in \mathbb{R}^n} \frac{||Ax||_1}{||x||_1}$$

is given by

$$||A||_1 = \max_{j=1,\ldots,n} \sum_{i=1}^n |A_{ij}|.$$

(b) Let $||\cdot||_2$ denote the Euclidean norm on $\mathbb{R}^n$ given by $||x||_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. For the matrix

$$A = \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}$$

compute the corresponding matrix operator norm $||A||_2$.

2. (10 pts)

Let A and B be $n \times n$ real matrices, A be non-singular, and satisfy the inequality

$$||A^{-1}||_2||B||_2 \leq q < 1,$$

where $||\cdot||_2$ is the matrix norm subordinate to the Euclidean 2-norm in $\mathbb{R}^n$.

(a) Show that the matrix $C = A + B$ is non-singular.

(b) Show that the iteration process $Ax_{j+1} = b - Bx_j$, $j = 0, 1, 2, \ldots$ converges for any initial value x_0 to the solution of the system $Cx = b$. Give an estimate for the Euclidean norm of the error $||x_j - x||_2$ in terms of q, where x is the true solution of the system and x_j is the jth approximation obtained from the iterative process.

3. (10 pts)

Use Householder reflections to find the QR-factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Check your answer by using matrix multiplication.

4. (10 pts)

Use the matrix exponential method to solve the following system of first order differential equations

$$\begin{cases} \dfrac{dx_1}{dt} &= x_1 + x_3 \\[2ex] \dfrac{dx_2}{dt} &= x_2 + x_3 \\[2ex] \dfrac{dx_3}{dt} &= x_2 + x_3 \end{cases}$$

with initial conditions $x_1(0) = 2,\ x_2(0) = 2,\ x_3(0) = 0.$

5. (10 pts)

Consider the Hilbert space $H = L^2(0, 2\pi)$ with the standard inner product $\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$.

Show that the orthogonal set $\{\sin(nx)\}_{n=1}^{\infty}$ is not a complete set in H.

6. (10 pts)

(a) Find the resolvent for the integral equation

$$u(x) - 6\int_0^1 xy\, u(y)\, dy = f(x), \quad x \in [0,1].$$

(b) Use Part (a) to solve the equation when $f(x) = x^2$.