

**1. (10 pts)**

- (a) Let A be an $n \times n$ real matrix and $\|\cdot\|_1$ denote the norm on $\mathbb{R}^n$ given by $\|x\|_1 = \sum_{i=1}^n |x_i|$ for any vector $x \in \mathbb{R}^n$. Prove that the matrix operator norm

$$\|A\|_1 = \max_{x \neq 0, x \in \mathbb{R}^n} \frac{\|Ax\|_1}{\|x\|_1}$$

is given by

$$\|A\|_1 = \max_{j=1, \dots, n} \sum_{i=1}^n |A_{ij}|.$$

- (b) Let $\|\cdot\|_2$ denote the Euclidean norm on $\mathbb{R}^n$ given by $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. For the matrix

$$A = \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}$$

compute the corresponding matrix operator norm $\|A\|_2$.



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2. (10 pts)

Let A and B be $n \times n$ real matrices, A be non-singular, and satisfy the inequality

$$\|A^{-1}\|_2 \|B\|_2 \leq q < 1,$$

where $\|\cdot\|_2$ is the matrix norm subordinate to the Euclidean 2-norm in $\mathbb{R}^n$.

- (a) Show that the matrix $C = A + B$ is non-singular.
- (b) Show that the iteration process $Ax_{j+1} = b - Bx_j$, $j = 0, 1, 2, \dots$ converges for any initial value x_0 to the solution of the system $Cx = b$. Give an estimate for the Euclidean norm of the error $\|x_j - x\|_2$ in terms of q , where x is the true solution of the system and x_j is the j th approximation obtained from the iterative process.



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**3. (10 pts)**

Use Householder reflections to find the QR -factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Check your answer by using matrix multiplication.



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**4. (10 pts)**

Use the matrix exponential method to solve the following system of first order differential equations

$$\begin{cases} \frac{dx_1}{dt} = x_1 + x_3 \\ \frac{dx_2}{dt} = x_2 + x_3 \\ \frac{dx_3}{dt} = x_2 + x_3 \end{cases}$$

with initial conditions $x_1(0) = 2$, $x_2(0) = 2$, $x_3(0) = 0$.



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**5. (10 pts)**

Consider the Hilbert space $H = L^2(0, 2\pi)$ with the standard inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x) dx$.

Show that the orthogonal set $\{\sin(nx)\}_{n=1}^{\infty}$ is not a complete set in H .



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**6. (10 pts)**

- (a) Find the resolvent for the integral equation

$$u(x) - 6 \int_0^1 xy u(y) dy = f(x), \quad x \in [0, 1].$$

- (b) Use Part (a) to solve the equation when $f(x) = x^2$.



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