

**1. (10 pts)**

Let H be a 3×3 Hilbert matrix:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

- (a) Find the decomposition $H = LDL^T$, where D is a 3×3 diagonal matrix and L is a 3×3 lower unit triangular matrix.
- (b) Give an example of a 3×3 symmetric matrix M which cannot be decomposed in the way described above.



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2. (10 pts) An $n \times n$ real matrix A is called *positive definite* if for any non-zero vector $x \in \mathbb{R}^n$ it follows that $x^T A x > 0$.

Let A be a positive definite matrix.

- (a) Prove that A is non-singular.
- (b) Does it follow from the definition above that A is symmetric? If yes, prove it. If no, give an example of a non-symmetric positive definite matrix (be sure to show explicitly that your matrix is positive definite).



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3. (10 pts) Let A be an $n \times n$ real symmetric matrix. Prove the following property: $\lim_{k \rightarrow \infty} A^k = O_{n \times n}$ if and only if $\rho(A) = \max_k |\lambda_k| < 1$, where λ_k ($k = 1, 2, \dots, n$) are the eigenvalues of the matrix A and $O_{n \times n}$ is the $n \times n$ zero matrix.

Hint: Use the spectral theorem for symmetric matrices.



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4. (10 pts) Assume $\{\phi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for $L^2(a, b)$. Define

$$\psi_{nm}(x, y) = \phi_n(x)\phi_m(y), \quad \text{for } (x, y) \in (a, b) \times (a, b).$$

Show that $\{\psi_{nm}\}_{n,m \in \mathbb{N}}$ forms an orthonormal basis for $L^2((a, b) \times (a, b))$.



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**5. (10 pts)**

- a) Find the eigenvalues and eigenfunctions of the integral operator

$$Ku(x) = \int_0^1 k(x, y) u(y) dy, \quad x \in [0, 1],$$

where

$$k(x, y) = \begin{cases} y(x-1), & 0 \leq x \leq y \leq 1 \\ x(y-1), & 0 \leq y \leq x \leq 1 \end{cases}.$$

- b) Find a spectral decomposition of K (that is, apply the Spectral Theorem to find a series representation for Ku).



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6. (8 pts) Show that the derivative operator, defined by

$$Lf = f',$$

where

$$L : (C[0, 1], \|\cdot\|_\infty) \longrightarrow (C[0, 1], \|\cdot\|_\infty),$$

with domain $D(L) = C^1[0, 1]$, is linear and unbounded.



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