



1. (10 pts)

Let H be a  $3 \times 3$  Hilbert matrix:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

(a) Find the decomposition  $H = LDL^T$ , where D is a  $3 \times 3$  diagonal matrix and L is a  $3 \times 3$  lower unit triangular matrix.

(b) Give an example of a  $3 \times 3$  symmetric matrix M which cannot be decomposed in the way described above.

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#1 4 of 16

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2. (10 pts) An  $n \times n$  real matrix A is called *positive definite* if for any non-zero vector  $x \in \mathbb{R}^n$  it follows that  $x^T A x > 0$ .

Let A be a positive definite matrix.

(a) Prove that A is non-singular.

(b) Does it follow from the definition above that A is symmetric? If yes, prove it. If no, give an example of a non-symmetric positive definite matrix (be sure to show explicitly that your matrix is positive definite).

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#1 6 of 16



**3.** (10 pts) Let A be an  $n \times n$  real symmetric matrix. Prove the following property:  $\lim_{k\to\infty} A^k = O_{n\times n}$  if and only if  $\rho(A) = \max_k |\lambda_k| < 1$ , where  $\lambda_k$  (k = 1, 2, ...n) are the eigenvalues of the matrix A and  $O_{n\times n}$  is the  $n \times n$  zero matrix.

Hint: Use the spectral theorem for symmetric matrices.

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applied-math-qe-i-june-2018-1e5bf
#1 8 of 16



4. (10 pts) Assume  $\{\phi_n\}_{n\in\mathbb{N}}$  is an orthonormal basis for  $L^2(a,b)$ . Define

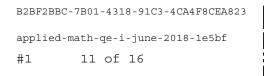
$$\psi_{nm}(x,y) = \phi_n(x)\phi_m(y), \text{ for } (x,y) \in (a,b) \times (a,b).$$

Show that  $\{\psi_{nm}\}_{n,m\in\mathbb{N}}$  forms an orthonormal basis for  $L^2((a,b)\times(a,b))$ .

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applied-math-qe-i-june-2018-1e5bf #1 10 of 16





## 5. (10 pts)

a) Find the eigenvalues and eigenfunctions of the integral operator

$$Ku(x) = \int_0^1 k(x, y) \ u(y) \ dy, \quad x \in [0, 1],$$

where

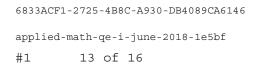
$$k(x,y) = \begin{cases} y(x-1), & 0 \le x \le y \le 1\\ x(y-1), & 0 \le y \le x \le 1 \end{cases}.$$

b) Find a spectral decomposition of K (that is, apply the Spectral Theorem to find a series representation for Ku).

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applied-math-qe-i-june-2018-1e5bf #1 12 of 16





6. (8 pts) Show that the derivative operator, defined by

$$Lf = f',$$

where

$$L: (C[0,1], \|\cdot\|_{\infty}) \longrightarrow (C[0,1], \|\cdot\|_{\infty}),$$

with domain  $D(L) = C^{1}[0, 1]$ , is linear and unbounded.

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applied-math-qe-i-june-2018-1e5bf #1 14 of 16

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#1 15 of 16



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applied-math-qe-i-june-2018-1e5bf
#1 16 of 16