**1. (10 pts)**

Let $H$ be a $3 \times 3$ Hilbert matrix:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

(a) Find the decomposition $H = LDL^T$, where $D$ is a $3 \times 3$ diagonal matrix and $L$ is a $3 \times 3$ lower unit triangular matrix.

(b) Give an example of a $3 \times 3$ symmetric matrix $M$ which cannot be decomposed in the way described above.

**2. (10 pts)** An $n \times n$ real matrix $A$ is called *positive definite* if for any non-zero vector $x \in \mathbb{R}^n$ it follows that $x^T A x > 0$.

Let $A$ be a positive definite matrix.

(a) Prove that A is non-singular.

(b) Does it follow from the definition above that A is symmetric? If yes, prove it. If no, give an example of a non-symmetric positive definite matrix (be sure to show explicitly that your matrix is positive definite).

**3. (10 pts)** Let $A$ be an $n \times n$ real symmetric matrix. Prove the following property: $\lim_{k\to\infty} A^k = O_{n\times n}$ if and only if $\rho(A) = \max_k |\lambda_k| < 1$, where $\lambda_k$ $(k = 1, 2, ...n)$ are the eigenvalues of the matrix $A$ and $O_{n\times n}$ is the $n \times n$ zero matrix.

Hint: Use the spectral theorem for symmetric matrices.

**4. (10 pts)** Assume $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(a,b)$. Define

$$\psi_{nm}(x,y) = \phi_n(x)\phi_m(y), \quad \text{for } (x,y) \in (a,b)\times(a,b).$$

Show that $\{\psi_{nm}\}_{n,m\in\mathbb{N}}$ forms an orthonormal basis for $L^2\left((a,b)\times(a,b)\right)$.

**5. (10 pts)**

a) Find the eigenvalues and eigenfunctions of the integral operator

$$Ku(x) = \int_0^1 k(x,y)\, u(y)\, dy, \quad x \in [0,1],$$

where

$$k(x,y) = \begin{cases} y(x-1), & 0 \le x \le y \le 1 \\ x(y-1), & 0 \le y \le x \le 1 \end{cases}.$$

b) Find a spectral decomposition of $K$ (that is, apply the Spectral Theorem to find a series representation for $Ku$).

**6. (8 pts)** Show that the derivative operator, defined by

$$Lf = f',$$

where

$$L : (C[0,1], \|\cdot\|_\infty) \longrightarrow (C[0,1], \|\cdot\|_\infty),$$

with domain $D(L) = C^1[0,1]$, is linear and unbounded.