Applied Math QE I Exam
August 2019

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

Consider the matrix decomposition:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & 3 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 4 & 2 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

(a) Label the columns of the leftmost matrix and the rows of the rightmost matrix, and use them to write A as a sum of outer products matrices;

(b) deduce from (a) the singular value decomposition of A (Justify);

(c) write down a matrix A_1 which is a best rank-one approximation of A;

(d) compute the condition number $\kappa_2(A)$.

2. (10 pts)

Let A be an $n \times n$ real matrix and $||\cdot||_2$ denote the Euclidean norm on $\mathbb{R}^n$ given by $||x||_2 := \sqrt{\sum_{i=1}^n |x_i|^2}$, for any vector $x \in \mathbb{R}^n$.

(a) Prove that the matrix operator norm

$$||A||_2 := \max_{x \neq 0, x \in \mathbb{R}^n} \frac{||Ax||_2}{||x||_2}$$

satisfies the following property: Given two $n \times n$ matrices A and B,

$$||AB||_2 \leq \|A\|_2 \|B\|_2.$$

(b) If A is an $n \times n$ invertible matrix, show that

$$\|A\|_2 \|A^{-1}\|_2 \geq 1.$$

(c) Recall that $\max_{i,j} |A_{ij}| \leq \|A\|_2$. Use this fact to show that, if X is an $n \times n$ real matrix with $\|X\|_2 < 1$, then $X^n \to 0$ as $n \to \infty$. Hence, conclude that in this case $A := I - X$ is invertible.

3. (10 pts) Assume A and E are $n \times n$ symmetric, positive definite matrices. Let α_n be the smallest eigenvalue of A and $\hat{\alpha}_n$ be the smallest eigenvalue of $A + E$. Show that

$$|\alpha_n - \hat{\alpha}_n| \leq \|E\|_2.$$

Hint: Use the minimum stretch characterization of the smallest eigenvalue for symmetric positive definite matrices.

4. (10 pts)

Find the second distributional derivative D^2u of the function

$$u(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

5. (10 pts)

Solve the following linear integral equation:

$$u(x) = \lambda K u(x) + f(x), \quad x \in [-1, 1]$$

where the integral operator K is given by

$$(Ku)(x) = \int_{-1}^{1} (x - y) u(y) dy.$$

Assume that the function $f(x)$ is piecewise continuous on $[-1, 1]$.

Investigate the existence and uniqueness of the solution for different values of the parameter λ.

6. (10 pts)

Show that the differential operator

$$Lu(x) = x^2 u''(x) + 2xu'(x) + u(x), x \in [0, 1],$$

subject to the periodic boundary conditions

$$u(0) = u(1), \;\; u'(0) = u'(1)$$

is self-adjoint under the standard inner product in $L^2[0, 1]$.

You need to show your derivations: in particular, stating that the operator is Sturm-Liouville will not be sufficient for full credit.