Applied Math QE I Exam
June 2019

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts) Consider the matrix decomposition:

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

(a) Label the columns of the leftmost matrix and the rows of the rightmost matrix, and use them to write A as a sum of outer products matrices;

(b) deduce from (a) the singular value decomposition of A (Justify);

(c) write down a matrix A_1 which is a best rank-one approximation of A;

(d) compute the condition number $\kappa_2(A)$.

2. (10 pts)

Find explicit constants $C_1, C_2 > 0$ such that

$$C_1 \|A\|_{max} \leq \|A\|_2 \leq C_2 \|A\|_{max}$$

for an arbitrary n-by-n matrix A, where $\|A\|_2$ is the operator norm corresponding to the Euclidean 2-norm on $\mathbb{R}^n$, and $\|A\|_{max}$ is the *max norm* $\max_{i,j} |A(i,j)|$. Aim for $C_1 = 1$ and $C_2 = n$ or at least $n^{3/2}$.

3. (10 pts)

Assume the matrix $A \in \mathbb{R}^{n \times n}$ is symmetric and positive semi-definite.

(a) Write down the "quadratic form definition" of what it means for A to be positive semi-definite.

(b) Assume $\lambda \in \mathbb{R}$ is an eigenvalue of A. Show that $\lambda \geq 0$.

(c) Recall that A admits an orthonormal basis of eigenvectors $\{v_1, \dots, v_n\}$, so that A is orthogonally diagonalizable via the orthoonal matix $Q = [v_1 \ \cdots \ v_n]$. In particular, given $u \in \mathbb{R}^n \setminus \{0\}$, we can write $u = \sum_{k=1}^{n}(u^T v_k)v_k = \sum_{k=1}^{n} c_k v_k = Qx$, where $x = [c_1 \cdots c_n]^T$. Use this fact to show that the Rayleigh quotient

$$\rho(A, u) := \frac{u^T A u}{u^T u}$$

for a vector $u \in \mathbb{R}^n \setminus \{0\}$, can be interpreted as a weighted average

$$\frac{1}{(\sum_{j=1}^{n} a_j)} \sum_{j=1}^{n} a_j \lambda_j$$

of the eigenvalues of A, with $a_j \geq 0$. Indeed, find the coefficients a_j.

4. (10 pts) Find the second distributional derivative D^2u of the function

$$u(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases} \qquad \text{for } x \in \mathbb{R}.$$

5. (10 pts) Find the eigenvalues and eigenfunctions of the integral operator

$$(Ku)(x) = \int_0^1 k(x,y)u(y)dy,$$

where $k(x,y) = \min\{x,y\}$, for $0 \le x \le 1$.

6. (10 pts) Consider the differential operator

$$Lu(x) = u''(x) + k^2 u(x), \;\; k \neq 0,$$

subject to the homogeneous boundary conditions

$$u'(0) = 0, \;\; u(\pi) = 0.$$

(1) Find the Green's function for this operator with the given boundary conditions.

(2) Using the Green's function from the previous step, solve the boundary value problem:

$$u''(x) + k^2 u(x) = f(x), \;\; u'(0) = 1, \;\; u(\pi) = 0.$$