Applied Math
Qual Exam Spring 2019
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You must **show your work clearly and justify everything** to receive credit.

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

Problem 1 [10 points] Consider the matrix decomposition:

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

(a) Label the columns of the leftmost matrix and the rows of the rightmost matrix, and use them to write A as a sum of outer products matrices;

(b) deduce from (a) the singular value decomposition of A (Justify);

(c) write down a matrix A_1 which is a best rank-one approximation of A;

(d) compute the condition number $\kappa_2(A)$.

(a) Write $a_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Then

$$A = -2a_1 b_1^T + 3a_2 b_2^T.$$

(b) Note that $a_1^T a_2 = b_1^T b_2 = 0$. So we let

$$u_2 := -\frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \text{ and } u_1 := \frac{a_2}{\|a_2\|} = \frac{1}{\sqrt{20}} \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

Also

$$v_2 := \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } v_1 := \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Then

$$A = 3\|a_2\|\|b_2\|u_1 v_1^T + 2\|a_1\|\|b_1\|u_2 v_2^T = 30u_1 v_1^T + 2\sqrt{30}u_2 v_2^T$$

In particular, by uniqueness, $\sigma_1 = 30$ and $\sigma_2 = 3\sqrt{30}$.

(c) Define $A_1 = 30u_1 v_1^T = 3a_2 b_2^T = 6 \begin{bmatrix} -4 & 2 \\ -2 & 1 \\ 0 & 0 \end{bmatrix}$

(d) Note A is full-rank, so $\kappa_2(A) = \frac{\sigma_1}{\sigma_2} = \frac{\sqrt{30}}{2}$.

Problem 2 [10 points] Find explicit constants $C_1, C_2 > 0$ such that

$$C_1\|A\|_{max} \le \|A\|_2 \le C_2\|A\|_{max}$$

for an arbitrary n-by-n matrix A, where $\|A\|_2$ is the operator norm corresponding to the Euclidean 2-norm on $\mathbb{R}^n$, and $\|A\|_{max}$ is the *max norm* $\max_{i,j}|A(i,j)|$. Aim for $C_1 = 1$ and $C_2 = n$, or at least $n^{3/2}$.

For $1 \le i, j \le n$:

$$|A(i,j)| = |e_i^T A e_j| \le \|A\|_2\|e_i\|\|e_j\| = \|A\|_2.$$

So $\|A\|_{max} \le \|A\|_2$.

On the other hand, if v is the direction of maximum stretch:

$$\|A\|_2 = \sigma = \|Av\| = \sqrt{\sum_{i=1}^n \left|\sum_{j=1}^n A(i,j)v(j)\right|^2} \le \|A\|_{max}\sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n |v_1(j)|\right)^2}$$
$$\le n^{3/2}\|A\|_{max}$$

where we used the fact that $|v(j)| \le 1$ for all j's.

Or using Cauchy-Schwarz:

$$\sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n |v(j)|\right)^2} \le \sqrt{\sum_{i=1}^n n \sum_{j=1}^n |v(j)|^2} = n.$$

Problem 3 [10 points] Assume the matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and positive semi-definite.

(a) Write down the "quadratic form definition" of what it means for A to be positive semi-definite.

(b) Assume $\lambda \in \mathbb{R}$ is an eigenvalue of A. Show that $\lambda \geq 0$.

(c) Recall that A admits an orthonormal basis of eigenvectors $\{v_1, \dots, v_n\}$, so that A is orthogonally diagonalizable via the orthoonal matix $Q = [v_1 \ \cdots \ v_n]$. In particular, given $u \in \mathbb{R}^n \setminus \{0\}$, we can write $u = \sum_{k=1}^n (u^T v_k)v_k = \sum_{k=1}^n c_k v_k = Qx$, where $x = [c_1 \cdots c_n]^T$. Use this fact to show that the Rayleigh quotient

$$\rho(A, u) := \frac{u^T A u}{u^T u}$$

for a vector $u \in \mathbb{R}^n \setminus \{0\}$, can be interpreted as a weighted average

$$\frac{1}{(\sum_{j=1}^n a_j)} \sum_{j=1}^n a_j \lambda_j$$

of the eigenvalues of A, with $a_j \geq 0$. Indeed, find the coefficients a_j.

(a) $x^T A x \geq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.
(b) There is $x \neq 0$ such that $Ax = \lambda x$. Then

$$0 \leq x^T A x = x^T(\lambda x) = \lambda \|x\|^2.$$

So $\lambda \geq 0$.
(c) Let $v_1, \dots, v_n$ be an orthonormal basis of eigenvectors for A with respective eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Let $Q = [v_1 \ \cdots \ v_n]$ be the orthogonal matrix corresponding to the ONB and $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Then $A = QDQ^T$. Given, $u \neq 0$, write $u = \sum_{k=1}^n (u^T v_k)v_k = \sum_{k=1}^n c_k v_k = Qx$, where $x = [c_1 \cdots c_n]^T$. Then

$$\rho(A, u) = \frac{x^T Q^T A Q x}{x^T Q^T Q x} = \frac{x^T D x}{x^T x} = \frac{\sum_{k=1}^n c_k^2 \lambda_k}{\sum_{k=1}^n c_k^2}.$$

So the coefficients are $a_k := c_k^2 = (u^T v_k)^2$.

Problem 4 [10 points] Find the second distributional derivative D^2u of the function

$$u(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases} \qquad \text{for } x \in \mathbb{R}.$$

By definition:

$$\begin{aligned} \langle D^2u, \varphi \rangle = \langle u, \varphi'' \rangle &= \int_{-\infty}^{\infty} u(x)\varphi''(x)dx = \int_{-1}^{1} (1 - |x|)\varphi''(x)dx \\ &= \int_{-1}^{0} (1 + x)\varphi''(x)dx + \int_{0}^{1} (1 - x)\varphi''(x)dx \\ &= (1 + x)\varphi'(x)\Big|_{-1}^{0} - \int_{-1}^{0} \varphi'(x)dx + (1 - x)\varphi'(x)\Big|_{0}^{1} + \int_{0}^{1} \varphi'(x)dx \\ &= \varphi(-1) + \varphi(1) - 2\varphi(0) = \langle \delta_{-1}, \varphi \rangle + \langle \delta_1, \varphi \rangle - 2\langle \delta_0, \varphi \rangle \end{aligned}$$

Then $D^2u = \delta_{-1} + \delta_1 - 2\delta_0$, where δ_{x_0} is a Dirac delta centered at x_0.

Problem 5 [10 points] Find the eigenvalues and eigenfunctions of the integral operator

$$(Ku)(x) = \int_0^1 k(x,y)u(y)dy,$$

where $k(x,y) = \min\{x,y\}$, for $0 \le x \le 1$.

$$(Ku)(x) = \int_0^1 k(x,y)u(y)dy = \int_0^x yu(y)dy + \int_x^1 xy(y)dy.$$

To find eigenvalues and eigenfunctions we need to solve $Ku = \lambda u$, ie.

$$\lambda u(x) = \int_0^x yu(y)dy + \int_x^1 xy(y)dy.$$

Differentiating,

$$\lambda u'(x) = \int_x^1 u(y)dy \qquad \text{and} \qquad \lambda u''(x) = -u(x)$$

The boundary conditions are $u(0) = 0$ and $u'(1) = 0$. To solve the boundary value problem $\lambda u'' + u = 0$ consider the equation

$$r^2 + \frac{1}{\lambda} = 0$$

• **Case 1:** $\lambda < 0$**.** Then $r_{1,2} = \pm\frac{1}{\sqrt{-\lambda}}$, and $u = C_1 e^{x/\sqrt{-\lambda}} + C_2 e^{-x/\sqrt{-\lambda}}$. To satisfy the boundary conditions we get $C_1 = C_2 = 0$.

• **Case 2:** $\lambda > 0$**.** Then $r_{1,2} = \pm\frac{i}{\sqrt{\lambda}}$, and $u = C_1 \cos\frac{x}{\sqrt{\lambda}} + C_2 \sin\frac{x}{\sqrt{\lambda}}$. Here $u(0) = 0$ implies $C_1 = 0$ and $u'(1) = 0$ implies

$$\frac{1}{\sqrt{\lambda}} = \frac{\pi}{2} + \pi k, \qquad k = 0, 1, 2, \dots$$

So

$$\lambda_k = \left(\frac{\pi}{2} + \pi k\right)^{-2} \qquad \text{and} \qquad u_k(x) = \sin\left(\left(\frac{\pi}{2} + \pi k\right) x\right)$$

Problem 6 [10 points] Consider the differential operator

$$Lu(x) = u''(x) + k^2 u(x), \ \ k \neq 0,$$

subject to the homogeneous boundary conditions

$$u'(0) = 0, \ \ u(\pi) = 0.$$

(1) Find the Green's function for this operator with the given boundary conditions.

(2) Using the Green's function from the previous step, solve the boundary value problem:

$$u''(x) + k^2 u(x) = f(x), \ \ u'(0) = 1, \ \ u(\pi) = 0.$$

(1) Solve the equation $u'' + k^2 u = 0$. Note that $r^2 + k^2 = 0$ implies $r = \pm ki$. Thus $u(x) = C_1 \cos kx + C_2 \sin kx$. Now find two solutions u_1, u_2 such that $u_1'(0) = 0$ and $u_2(\pi) = 0$. We get $u_1(x) = \cos kx$ and $u_2(x) = \sin kx$. Build the Green's function as

$$G(x,y) = \begin{cases} \frac{1}{W(y)} u_1(x) u_2(y), & 0 \leq x < y \leq \pi \\ \frac{1}{W(y)} u_1(y) u_2(x), & 0 \leq y < x \leq \pi \end{cases}$$

where

$$W(y) = \begin{vmatrix} \cos ky & \sin ky \\ -k \sin ky & k \cos ky \end{vmatrix} = k$$

Thus

$$G(x,y) = \begin{cases} \frac{1}{k} \cos kx \sin ky, & 0 \leq x < y \leq \pi \\ \frac{1}{k} \sin kx \cos ky, & 0 \leq y < x \leq \pi \end{cases}$$

(2) The solution will look like

$$u(x) = \int_0^\pi G(x,y) f(y) dy + C_1 u_1(x) + C_2 u_2(y)$$

where $C_1 = \frac{u(\pi)}{u_1(\pi)} = 0$ and $C_2 = \frac{u'(0)}{u_2'(0)} = \frac{1}{k}$.