Applied Mathematics Qualifying Exam, August 2020

Problem 1. Let $A \in \mathbb{R}^{m\times n}$. Show that $A^T A$ is nonsingular if and only if A has full column rank.

Problem 2. Suppose you are working with rank-one matrices $A = ab^T \in \mathbb{R}^{m\times n}$ and $B = cd^T \in \mathbb{R}^{n\times p}$ with $a \in \mathbb{R}^m$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}^p$.

(a) Assuming the vectors a, b, c and d are all known, describe a memory-efficient method for storing A and B. (Storing all entries of A requires keeping $O(mn)$ floating point numbers in memory. Your scheme should require the storage of significantly fewer floating point numbers.)

(b) Describe a computationally efficient method for computing the product $C = AB$. C should be stored in memory in the same format as A and B. Your algorithm should require $\min\{m, p\} + O(n)$ floating point operations.

Problem 3. Let $A \in \mathbb{R}^{m\times n}$ have singular value decomposition $A = U\Sigma V^T$ where $U \in \mathbb{R}^{m\times r}$ and $V \in \mathbb{R}^{n\times r}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{r\times r}$ is nonsingular. Write a singular value decomposition for each of the following. (A^+ represents the Moore–Penrose pseudoinverse.)

(a) A^+

(b) A^+A

(c) A^+AA^+

(d) A^+AA^TA

Problem 4. Let $C[a, b]$ be the vector space of continuous functions on the interval $[a, b]$. Prove that $C[a, b]$ is a subspace of $C[c, d]$ if $[c, d] \subset [a, b]$.

Problem 5. Let H be a Hilbert space. Consider a bounded linear functional $f : H \to \mathbb{R}$.

a) State the Riesz representation theorem for the functional f.

b) Prove the Riesz representation theorem when $H = \mathbb{R}^n$.

Problem 6. Let H be a Hilbert space. Let (T_n) be a convergent sequence of self-adjoint linear operators on H. Prove that the limit of (T_n) is also self-adjoint.