Applied Mathematics Qualifying Exam, June 2020

Problem 1. Derive the normal equations for the least squares problem

$$\min_x \|Ax - b\|_2.$$

Problem 2. Let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ -2 & 2 & 0 \end{bmatrix}.$$

Find each of the following.

(a) The singular values of A.

(b) The operator 2-norm, $\|A\|_2$, of A.

(c) The Moore-Penrose pseudoinverse, A^+, of A.

Hint: Consider $A^T A$.

Problem 3. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

(a) Define the induced operator norm, $\|\cdot\|$, on $\mathbb{R}^{n\times n}$.

(b) Prove that $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$.

Problem 4. Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator. Let $N(T)$ be the null space of T and $R(T^*)$ be the range of the adjoint operator of T. Prove that $N(T) = (R(T^*))^\perp$ (i.e. the orthogonal complement of $R(T^*)$.)

Problem 5. Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator. T is called unitary if $TT^* = T^*T = I$, where T^* is the adjoint operator of T. Prove that if T is unitary, then all eigenvalues of T have modulus 1 and eigenvectors associated with different eigenvalues are orthogonal.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Prove that the derivative of the distribution generated by f is equal to the distribution generated by f'.