Applied Math
Qual Exam Fall 2021
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Problem 1 [10 points] Let $A \in \mathbb{R}^{N\times N}$, with $\operatorname{rank}(A) = 1$.

(a) Show that A can be written as an outer product ab^T, with $a, b \in \mathbb{R}^N \setminus \{0\}$.

(b) Give an interpretation of the fundamental subspaces of A, the kernel $\operatorname{Ker}(A)$ and the range $\operatorname{Ran}(A)$, in terms of the vectors a and b in (a).

(c) Find the singular value decomposition (SVD) of A. How many non-zero singular values are there?

Problem 2 [10 points] Let $A \in \mathbb{R}^{N\times N}$. Recall that the operator norm of A is

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|},$$

where $\|x\|$ is the usual Euclidean norm.

(a) Show that if $A^T = A$, then $\|A\| = \max_{j=1,\dots,N} |\lambda_j(A)|$, where $\lambda_j(A)$ denotes the eigenvalues of A. Hint: Use the Spectral Theorem for symmetric matrices.

(b) Suppose A is positive definite. Write and justify a formula for the condition number of A in terms of its eigenvalues.

Problem 3 [10 points] Assume $A \in \mathbb{R}^{n\times n}$ is symmetric and $v \in \mathbb{R}^n \setminus \{0\}$. Set $\hat{A} := A + vv^T$. Let $\alpha_1 \leq \cdots \leq \alpha_n$ and $\hat{\alpha}_1 \leq \cdots \leq \hat{\alpha}_n$ be the eigenvalues for A and $\hat{A}$ respectively. Use the Courant-Fischer Theorem to show that, for $j = 1, \dots, n$,

$$\hat{\alpha}_j \leq \alpha_j + \|v\|^2,$$

where $\|v\|$ is the norm of v.
Hint:
Courant-Fischer Theorem: Let H be an $N \times N$ real symmetric matrix. Consider the eigenvalues ordered so that $\lambda_1(H) \leq \cdots \leq \lambda_N(H)$. Then:

$$\lambda_k(H) = \min_{S\in\mathcal{W}_k} \max_{u\in S\setminus\{0\}} \frac{u^T H u}{u^T u} = \max_{T\in\mathcal{W}_{N-k+1}} \min_{u\in T\setminus\{0\}} \frac{u^T H u}{u^T u} \tag{1}$$

where $\mathcal{W}_j := \{U \subset \mathbb{R}^N : U$ *is a linear subspace of* $\mathbb{R}^N$ *and* $\dim U = j\}$.

Problem 4 [10 points] Show that for all x, y in a real inner product space

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2.$$

For a complex inner product space, find a similar expression for (x, y) in terms of norms of combinations of x and y.

Problem 5 [10 points] Let H be a Hilbert space and $f : H \to \mathbb{R}$ be a linear functional. Show that f is continuous if and only if it is bounded.

Problem 6 [10 points] Let H be a Hilbert space. Show that if $A : H \to H$ is a self-adjoint operator, then all its eigenvalues are real, and eigenvectors belonging to different eigenvalues are orthogonal.