Applied Math
Qual Exam Spring 2021
Dinh-Liem Nguyen
Pietro Poggi-Corradini

Name:____________________

You must **show your work clearly and justify everything** to receive credit.

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

Problem 1 [10 points] Let $A \in \mathbb{R}^{N\times N}$, with $\text{rank}(A) = 1$.

(a) Show that A can be written as an outer product ab^T, with $a, b \in \mathbb{R}^N \setminus \{0\}$.

Since $\text{rank}(A) = 1$, we have $A \neq 0$. Let b^T be a non-zero row of A. Then, every row of A is of the form $a_j b^T$ for some real number $a_j \in \mathbb{R}$, $j = 1, \dots, N$. Let $a := [\cdots a_j \cdots]^T$. Then $A = ab^T$, and $a \neq 0$ again because $\text{rank}(A) = 1$.

(b) Give an interpretation of the fundamental subspaces of A, the kernel $\text{Ker}(A)$ and the range $\text{Ran}(A)$, in terms of the vectors a and b in (a).

Write

$$A = ab^T = \begin{bmatrix} \vdots & & \vdots \\ b_1 a & \cdots & b_n a \\ \vdots & & \vdots \end{bmatrix}$$

Then, we see that the column space of A is spanned by the vector a.

Also, since $a \neq 0$, $Ax = ab^T x = 0$, if and only if $b^T x = 0$. Therefore, $\ker(A)$ is the orthogonal complement of the line spanned by b.

(c) Assume that $b^T a \neq 0$ and compute the eigenvalues and eigenvectors of A in terms of a and b, in this case.

By Rank-Nullity, $\dim \ker(A) = N-1$. So $\lambda = 0$ is an eigenvalue of multiplicity $N - 1$. Moreover, every vector $u \in \langle b \rangle^\perp$ is an eigenvector for it.

However, since $b^T a \neq 0$, $a \notin \langle b \rangle^\perp$. And, if $x = a$, then

$$Ax = ab^T a = (b^T a)a.$$

So a is an eigenvector with eigenvalue $\lambda = b^T a \neq 0$.

Problem 2 [10 points] Let $A \in \mathbb{R}^{N\times N}$. Recall that the operator norm of A is

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|},$$

where $\|x\|$ is the usual Euclidean norm.

(a) Show that if $A^T = A$, then $\|A\| = \max_{j=1,\ldots,N} |\lambda_j(A)|$, where $\lambda_j(A)$ denotes the eigenvalues of A. Hint: Use the Spectral Theorem for symmetric matrices.

Since A is symmetric, the spectral theorem implies that A admits an orthonormal basis of eigenvectors $\{u_1, \ldots, u_N\}$ (here $Au_j = \lambda_j u_j$). Expand x in this basis as

$$x = \sum_{j=1}^{N} (u_j^T x) u_j.$$

Then,

$$\frac{\|Ax\|^2}{\|x\|^2} = \frac{\left\|\sum_{j=1}^N (u_j^T x)\lambda_j u_j\right\|}{\left\|\sum_{j=1}^N (u_j^T x) u_j\right\|} = \frac{\sum_{j=1}^N (u_j^T x)^2 \lambda_j^2}{\sum_{j=1}^N (u_j^T x)^2}$$

The right hand-side is a weighted average of the numbers $\{\lambda_j^2\}_{j=1}^N$. Hence, the maximum is attained by $\lambda_{\max}^2$. The conclusion follows by taking square roots.

(b) Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, does not satisfy the conclusion in (a).

Since A is triangular, the eigenvalues are given by the diagonal elements. So $\lambda_{\max} = 1$ in this case. However, if $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$Ax = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$\frac{\|Ax\|}{\|x\|} = \frac{\sqrt{5}}{\sqrt{2}} > 1.$$

Problem 3 [10 points] Assume $A, E \in \mathbb{R}^{n\times n}$ are symmetric. Set $\hat{A} := A + E$. Let $\alpha_1 \le \cdots \le \alpha_n$ and $\hat{\alpha}_1 \le \cdots \le \hat{\alpha}_n$ be the eigenvalues for A and $\hat{A}$ respectively. Use the Courant-Fischer Theorem to show that, for $j = 1, \ldots, n$,

$$\hat{\alpha}_j \le \alpha_j + \|E\|,$$

where $\|E\|$ is the operator norm of E.

Hint:

Courant-Fischer Theorem: Let H be an $N \times N$ real symmetric matrix. Consider the eigenvalues ordered so that $\lambda_1(H) \le \cdots \le \lambda_N(H)$. Then:

$$\lambda_k(H) = \min_{S\in\mathcal{W}_k} \max_{u\in S\setminus\{0\}} \frac{u^T H u}{u^T u} = \max_{T\in\mathcal{W}_{N-k+1}} \min_{u\in T\setminus\{0\}} \frac{u^T H u}{u^T u} \tag{1}$$

where $\mathcal{W}_j := \{U \subset \mathbb{R}^N : U$ is a linear subspace of $\mathbb{R}^N$ and $\dim U = j\}$.

By the Courant-Fischer Theorem,

$$\hat{\alpha}_j = \min_{S\in\mathcal{W}_j} \max_{u\in S\setminus\{0\}} \frac{u^T \hat{A} u}{u^T u} = \min_{S\in\mathcal{W}_j} \max_{u\in S\setminus\{0\}} \left(\frac{u^T A u}{u^T u} + \frac{u^T E u}{u^T u} \right).$$

Further,

$$\max_{u\in S\setminus\{0\}} \left(\frac{u^T A u}{u^T u} + \frac{u^T E u}{u^T u} \right) \le \max_{u\in S\setminus\{0\}} \frac{u^T A u}{u^T u} + \max_{u\in S\setminus\{0\}} \frac{u^T E u}{u^T u},$$

and

$$\max_{u\in S\setminus\{0\}} \frac{u^T E u}{u^T u} \le \max_{u\ne 0} \frac{u^T E u}{u^T u} = \|E\|.$$

So

$$\hat{\alpha}_j \le \min_{S\in\mathcal{W}_j} \left(\max_{u\in S\setminus\{0\}} \frac{u^T A u}{u^T u} + \|E\| \right) = \min_{S\in\mathcal{W}_j} \max_{u\in S\setminus\{0\}} \frac{u^T A u}{u^T u} + \|E\| = \alpha_j + \|E\|.$$

Problem 4 [10 points] Consider the following function

$$f(x) = \begin{cases} 0, & x < 0, \\ 2, & 0 \le x < 2, \\ 3, & x \ge 2. \end{cases}$$

Find the distributional derivative of f.

It is obvious that f is a locally integrable function. Let F be the distribution generated by f. For $\phi \in \mathcal{D}$, we have

$$\begin{aligned}(F', \phi) = -(F, \phi') = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx &= -2\int_0^2 \phi'(x)dx - 3\int_2^{\infty} \phi'(x)dx \\ &= \phi(2) + 2\phi(0) \\ &= (\delta_2, \phi) + 2(\delta, \phi) = (\delta_2 + 2\delta, \phi)\end{aligned}$$

Therefore $F' = \delta_2 + 2\delta$.

Problem 5 [10 points] Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator. Let $R(T)$ be the range of T and $N(T^*)$ be the null space of T^*. Prove that $(R(T))^\perp = N(T^*)$ (i.e. the orthogonal complement of $R(T)$.)

Let $u \in (R(T))^\perp$. Then $u \perp R(T)$ or $(u, Tv) = 0$ for all $v \in H$. This implies that $(T^*u, v) = 0$ for all $v \in H$, that means that $T^*u = 0$ or $u \in N(T^*)$.

Let $x \in N(T^*)$. Then $T^*x = 0$ or $(T^*x, v) = 0$ for all $v \in H$. This implies that $(x, Tv) = 0$ for all $v \in H$ or x belongs to $(R(T))^\perp$.

Problem 6 [10 points] Let H be a Hilbert space. A bounded linear operator $A : H \to H$ is called *normal* if it commutes with its adjoint: $AA^* = A^*A$. Show that if B, C are commuting self-adjoint operators on H, then $B + iC$ is normal.

We have from the assumption that $B = B^*$, $C = C^*$ and $BC = CB$. Let $u, v \in H$. We compute

$$((B + iC)u, v) = (Bu + iCu, v) = (u, Bv) + (u, -iCv) = (u, (B - iC)v)$$

Thus $(B + iC)^* = B - iC$. This implies that

$$(B + iC)^*(B + iC) = (B - iC)(B + iC) = B^2 + iBC - iCB + C^2 = B^2 + C^2$$

Similarly we also have $(B + iC)(B + iC)^* = B^2 + C^2$. Thus $B + iC$ is normal