Applied Mathematics Qualifying Exam, August 2022

Problem 1. Consider a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. Let $\|A\|_\infty$ be the matrix norm of A induced by the max norm $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ for $x \in \mathbb{R}^n$. Prove that

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

Problem 2. Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}$. Prove that the nonzero eigenvalues of AB and BA are the same.

Problem 3. Let V be a subspace of $\mathbb{R}^n$ and $u_1, u_2, \ldots, u_k$ be an orthonormal basis of V. Let $P : \mathbb{R}^n \to V$ be the orthogonal projection onto V. Prove that

$$Px = \sum_{j=1}^{k} (x \cdot u_j) u_j.$$

Problem 4. Find the second distributional derivative t'' of the function

$$t(x) = \begin{cases} 1 - x + x^3, & |x| < 1 \\ 1, & |x| \ge 1. \end{cases}$$

Problem 5. Solve the following linear integral equation:

$$u(x) = \lambda K u(x) + f(x), \;\; x \in [-1, 1]$$

where the integral operator K is given by

$$(Ku)(x) = \int_{-\pi}^{\pi} \sin(x+y) u(y) dy.$$

Characterize the existence and uniqueness of the solution for all values of the parameter λ.

Problem 6. Find the Green's function for the equation:

$$u'' - 9u = f(x),$$

with the boundary conditions:

$$u'(0) = 0, \;\; u(1) = 0.$$