Applied Mathematics Qualifying Exam, June 2022

INSTRUCTIONS

This is a closed-book exam. No written material, electronic tools, or communication with others is permitted. You have three hours to work and 20 minutes to upload solutions.

Four questions correctly solved (up to minor errors) will earn a pass on this exam. Parts of questions may in some cases combine to count as one full problem.

Do not write your name or any other identifying information on your work.

Submitting your work:

- Find the email from Crowdmark Mailer with subject "Graduate Program 2021–2022 – New Assignment: Applied Math" in the subject line. Follow steps given.
- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to sarahrez@ksu.edu.
- Before leaving the library, give all hard copies to the proctor.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

a) Find a singular value decomposition of A.
b) Find the Frobenius norm $\|A\|_F$ and the norm $\|A\|_2$ induced by the vector (Euclidean) 2-norm.

Problem 2. An $n \times n$ matrix A is called positive definite if $Ax \cdot x > 0$ for all nonzero vectors x in $\mathbb{R}^n$. Prove that a symmetric matrix A is positive definite if and only if all its eigenvalues are positive.

Problem 3. Consider a matrix A in $\mathbb{R}^{m\times n}$. Let $A^+ \in \mathbb{R}^{n\times m}$ be the (Moore-Penrose) pseudo-inverse of A. Define the transformation $P : \mathbb{R}^m \to \mathbb{R}^m$ by $Px = AA^+x$. Prove that P is the orthogonal projection onto range(A).

Problem 4. Find the second distributional derivative t'' of the function

$$t(x) = \begin{cases} \sin \pi x + x^2 - 1, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

Problem 5. Solve the following linear integral equation:

$$u(x) = \lambda K u(x) + f(x), \quad x \in [-1, 1]$$

where the integral operator K is given by

$$(Ku)(x) = \int_{-1}^{1} (xy - y^2) u(y)\, dy.$$

Characterize the existence and uniqueness of the solution for all values of the parameter λ.

Problem 6. Find the Green's function for the equation:

$$u'' + 4u = f(x),$$

with the boundary conditions:

$$u(0) = 0, \;\; u'(1) = 0.$$