Applied Mathematics QE I Exam
August 2023

Name . . . .

KSU Email . . . .

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. **(10 pts)** Let $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times m}$. Show that the nonzero eigenvalues of the matrices AB and BA are the same.

2. **(10 pts)** Let V be a subspace of $\mathbb{R}^n$ and $u_1, u_2, \ldots, u_k$ be an orthonormal basis of V. Define the mapping $P : \mathbb{R}^n \to V$ as

$$Px = \sum_{j=1}^{k} (x \cdot u_j) u_j.$$

For every $x \in \mathbb{R}^n$, prove that

$$\|Px - x\|_2 = \inf_{y \in V} \|y - x\|_2.$$

3. **(10 pts)** Let $A = [a_{ij}]$ be a matrix in $\mathbb{R}^{n\times n}$ and consider $\|\cdot\|_1$ to be the operator norm induced by vector norm $|\cdot|_1$ in $\mathbb{R}^n$. Prove that

$$\|A\|_1 = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|a_{ij}|\right).$$

4. **(10 pts)** Let n be a positive integer and

$$V_n = \text{span}\{e^{ijx}, -n \le j \le n\}.$$

Show that $\cup_{n=1}^{\infty} V_n$ is dense in $L^2(0, 2\pi)$.

5. **(10 pts)** Let f be a locally integrable function and $\mathcal{D}$ be the set of test functions. Define

$$(T, \phi) = \int_{\mathbb{R}} f(x)\phi(x)dx, \quad \text{for all } \phi \in \mathcal{D}.$$

Prove that T is a distribution.

6. **(10 pts)** Let H be a Hilbert space and $L : H \to H$ be a continuous linear operator. Prove that if $\|L\| < 1$, then the operator $I - L$ is invertible. Find the inverse of $I - L$ in this case.