Applied Math
Qual Exam Spring 2023

Name:____________________

You must **show your work clearly and justify everything** to receive credit.

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

Problem 1 [10 points] Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$ and assume that $A \in \mathbb{R}^{m\times n}$. Prove that if $\operatorname{rank}(A) = n$ then $\|x\|_A := \|Ax\|$ defines a vector norm on $\mathbb{R}^n$.

Problem 2 [10 points] Let V be a subspace of $\mathbb{R}^n$ and $u_1, u_2, \ldots, u_k$ be an orthonormal basis of V. Define the mapping $P : \mathbb{R}^n \to V$ as

$$Px = \sum_{j=1}^{k} (x \cdot u_j) u_j.$$

For every $x \in \mathbb{R}^n$, prove that

$$\|Px - x\|_2 = \inf_{y \in V} \|y - x\|_2.$$

Problem 3 [10 points] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix. Prove that there is a symmetric and positive definite matrix $B \in \mathbb{R}^{n\times n}$ such that $A = B^2$.

Problem 4 [10 points] Let H be a Hilbert space and $L : H \to H$ be a continuous linear operator. Show that L maps bounded sets to bounded sets in H.

Problem 5 [10 points] Let $\mathcal{D}$ be the set of test functions (smooth functions with compact support). Prove that, for any distribution F, the functional T defined by

$$(T, \phi) = -(F, \phi'), \quad \text{for all } \phi \in \mathcal{D}$$

is a distribution.

Problem 6 [10 points] Let H be a Hilbert space and $L : H \to H$ be a continuous linear operator. Prove that if $\|L\| < 1$, then the equation

$$u - Lu = f$$

has a unique solution $u \in H$ for any given right hand side $f \in H$.