Applied Math QE1

05/30/2024

Name: ______________________________

KSU Email: ______________________________

Instructions: You have three hours to do your work and fifteen minutes to upload the solutions.

You must **show your work clearly and justify everything** to receive credit.

0) Four full problems correctly done will earn a pass. Parts of several problems may combine to count for a full problem.

1) Do not write your name on any of the pages.

2) This is a closed book exam: no books, no notes, no calculators etc. Only plain papers and pens should be on your table.

3) You must have your camera on showing your hands and, if possible, at least part of your faces during the exam.

4) After you receive the exam online by email, if you want you can print it or you can keep it open on your laptop or cellphone. After that you are allowed to use your computer or any electronic device during the exam only to read the exam and later to upload it on Crowdmark. Also you can communicate to the examiner via private chat in Zoom.

5) In case you lose internet connection at some point, you can continue your exam, however the examiners might consider to have an oral reexamination with you where you would need to explain steps in your work. You can be asked additional questions. In case the loss of connection is long, you can contact the examiner.

6) The exam is supposed to take 3 hours not counting the time of printing or accessing the problems and uploading your test. If you want you can have a bit of extra time, but the exam must be uploaded to Crowdmark no less than 10 minutes after the exam

Submitting your work:

- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to tinaande@ksu.edu.

Problem 1 [10 points] A projector is a square matrix P that satisfies $P^2 = P$.
a) Prove that if P is a projector, then $null(I - P) = range(P)$.
b) Prove that if P is a projector, then $range(P) \cap null(P) = \{0\}$.

Problem 2 [10 points] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix. Prove that there is a symmetric and positive definite matrix $B \in \mathbb{R}^{n\times n}$ such that $A = BB^T$.

Problem 3 [10 points] Let $A \in \mathbb{R}^{n\times n}$ and assume that $\|A\|_p < 1$ for some $p \geq 1$ ($\|\cdot\|_p$ is the operator norm induced by the vector p-norm). Prove that the matrix $I_n - A$ is nonsingular and find its inverse.

Problem 4 [10 points] Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal set in Hilbert space H. Prove that for any $u \in H$ the series $\sum_{n=1}^{\infty} |(u, e_n)|^2$ is convergent and that

$$\sum_{n=1}^{\infty} |(u, e_n)|^2 \leq \|u\|^2.$$

Problem 5 [10 points] Let $f : \mathbb{R} \to \mathbb{C}$ be a locally integrable function and $\mathcal{D}$ be the set of test functions (smooth functions with compact support in $\mathbb{R}$). Let n be a positive integer, define $T : \mathcal{D} \to \mathbb{C}$ as

$$(T, \phi) = \int_{-\infty}^{\infty} e^{inx} \frac{d^n \phi(x)}{dx^n} f(x) dx \quad \text{for all } \phi \in \mathcal{D}.$$

Prove that the functional T is well-defined and that T is a distribution.

Problem 6 [10 points] Let H be a Hilbert space and $L : H \to H$ be a continuous linear operator with closed range. Prove that the equation $Lu = f$ has a solution if and only if f is orthogonal to the null space of the adjoint operator L^*.