# Applied Mathematics Practice Qualifying Exam

**Problem 1.** Let $A \in \mathbb{R}^{n\times n}$ be nonsingular.

(a) Describe the general numerical solution technique for the linear system of equations $Ax = b$.

(b) Explain both *how* to modify this technique if $A$ is symmetric positive definite and *why* it is helpful to do so.

**Problem 2.** Let $A \in \mathbb{R}^{m \times n}$. Explain how to solve the problem

$$\min_{\operatorname{rank}(B) \leq k} \|A - B\|_2$$

where $\|\cdot\|_2$ represents the operator 2-norm.

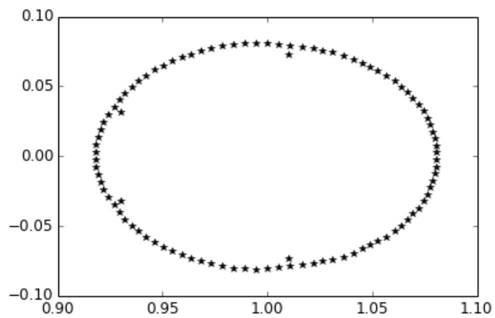
**Problem 3.** Let $A, B \in \mathbb{R}^{100\times 100}$ be matrices of the forms

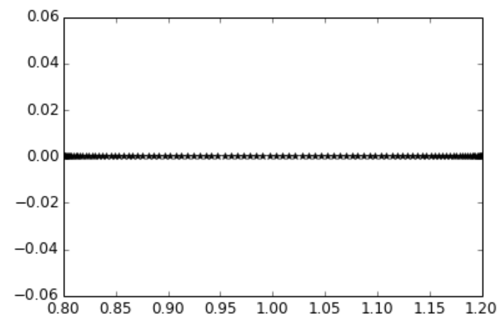
$$A = \begin{bmatrix} 1 & 0.1 & & & & \\ 0.1 & 1 & 0.1 & & & \\ & 0.1 & 1 & 0.1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0.1 & 1 & 0.1 \\ & & & & 0.1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0.1 & & & & \\ & 1 & 0.1 & & & \\ & & 1 & 0.1 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 0.1 \\ & & & & & 1 \end{bmatrix}$$

respectively. The figures below display the spectra of two matrices $\tilde{A}, \tilde{B} \in \mathbb{R}^{100\times 100}$, with the properties that

$$\|A - \tilde{A}\|_\infty \le 10^{-10} \quad \text{and} \quad \|B - \tilde{B}\|_\infty \le 10^{-10}.$$

Determine which figure corresponds to each matrix. Explain your reasoning.



**Problem 4.** Let $L$ be a self-adjoint bounded linear operator on a Hilbert space. Show that if $\mu$ and $\eta$ are distinct eigenvalues of $L$ with eigenvectors $u$ and $v$ respectively, then $u$ is orthogonal to $v$.

**Problem 5.** Consider the family of linear operators, parameterized by $\lambda \in \mathbb{R}$,

$$L_\lambda = I - \lambda K$$

on the Hilbert space $L^2([0,1])$, where

$$(Ku)(x) = \int_0^1 e^{x-y} u(y)\, dy.$$

(a) Prove that the equation $L_\lambda u = f$ is uniquely solvable when $\lambda \neq 1$.

(b) Determine necessary and sufficient conditions on $f$ for solvability when $\lambda = 1$.

**Problem 6.** Find the Green's function for the differential operator

$$Lu = -u'', \qquad u(0) = u'(1) = 0.$$