Topology Qualifying Examination
22 August 2015

You have been given a sheet with a code number. Put your name on this sheet. Do not put your name on any pages containing your solution to the exam. Put your code number at the upper right corner of the pages you turn in with your solutions.

You have been supplied with adequate blank paper. Please begin each problem on a new page, and order your pages by question number when you turn in the exam. Attempt all problems. Except when a definition or statement of a theorem is asked for, you should justify all of your answers.

Turn the pages with the code number in separately from your exam. Be sure that your code number is in the upper right corner of each page you turn in, and that the pages are ordered in order of question number.

1. Consider the finite topological space $X = \{0, x, 1\}$ with topology $\{\emptyset, \{0\}, \{1\}, X\}$.

   (a) Which of the following topological properties does this space exhibit: compactness, connectedness, path connectedness, contractibility, T_0-ness, Hausdorffness, metrizability?

   (b) Prove that if Y is a topological space, there is a bijection between the set of continuous functions from Y to X and the set
   $$\{(U, V) \mid U \text{ and } V \text{ are disjoint open subsets of } X\}.$$

2. A topological space X has the *fixed point property* if every continuous map $f\colon X \to X$ has at least one fixed point (i.e., there exists $x_f \in X$ such that $f(x_f) = x_f$).

   (a) Show that the sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ does not have the fixed point property.

   (b) Suppose that X has the fixed point property. Show that X is T_0, that is, for each pair of distinct points in X, at least one of the points has an open neighborhood which does not contain the other point (Hint: suppose otherwise and define a map $f\colon X \to X$ without fixed points).

3. (a) Define what it means for a topological space to be connected and to be path-connected.

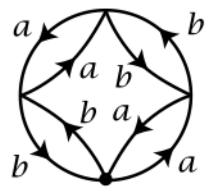
   (b) Prove that if a topological space is path-connected then it is also connected.

4. A set of sets $\mathcal{C}$ is said to have :*the finite intersection property* if whenever $\{C_1, \ldots C_n\} \subset \mathcal{C}$ is a finite subset of $\mathcal{C}$, $\cap_{j=1}^n C_j$ is non-empty. For a given topological space X, prove that the following statements are equivalent.

   - X is compact.
   - For each set $\mathcal{C}$ of *closed* subsets of X, $\mathcal{C}$ has the finite intersection property if and only if $\cap_{C \in \mathcal{C}} C$ is non-empty.

5. (a) Define what a deck transformation (of a covering space) is, and define what a normal covering space is.

   (b) Consider the topological space $X = S^1 \vee S^1$. Describe the group of deck transformations of the following covering space of X.



6. Find the homology groups of the 3-torus, $S^1 \times S^1 \times S^1$.