Topology Qualifying Examination
9 June 2015

You have been supplied with adequate blank paper. Please begin each problem on a new page, and order your pages by question number when you turn in the exam. Attempt all problems. Except when a definition or statement of a theorem is asked for, you should justify all of your answers.

1. Consider the finite topological space $X = \{0, x, 1\}$ with topology $\{\emptyset, \{1\}, \{x, 1\}, X\}$.
   (a) Which of the following topological properties does this space exhibit: compactness, connectedness, path connectedness, separability, contractibility, Hausdorffness, metrizability?
   (b) Prove that if Y is a topological space, there is a bijection between the set of continuous functions from Y to X and the set of nested pairs of open sets $U \subset V$ in Y.

2. (a) Define what it means for a topological space to be compact.
   (b) State Tychonoff's Theorem.
   (c) Prove the special case of Tychonoff's Theorem for two topological spaces.

3. Let Y be the space obtained from the torus $S^1 \times S^1$ by removing an open disk.

   Find the homology groups of Y.

4. Let $S^1 = \{z \in \mathbb{C} \mid \parallel z \parallel = 1\}$. Let $\iota_i : S^1 \to D_i^2$ be the inclusions of S^1 onto the boundaries of two disjoint copies of the unit disk $D^2 = \{z \in \mathbb{C} \mid \parallel z \parallel \leq 1\}$. Finally, let $X_{n,m} = S^1 \coprod D_1^2 \coprod D_2^2 / \sim$, where $\sim$ is the equivalence relation generated by $z \sim \iota_1(z^n)$ and $z \sim \iota_2(z^m)$ for n and m integers, for all $z \in S^1$.
   (a) Prove that $X_{n,m}$ is path-connected.
   (b) Compute the fundamental group $\pi_1(X_{n,m}, *)$ in terms of n and m, where $*$ is an arbitrary base point.

5. Let $X = S^1 \times S^1 \times S^1$, and let $Y = \mathbb{R}P^2 \times \mathbb{R}P^2$.
   (a) Define what it means for a map to be nulhomotopic.
   (b) Show that any map $f \colon Y \to X$ is nulhomotopic.

6. Prove that if X and Y are path connected spaces, and $A \subset X$ and $B \subset Y$ are proper subsets, then the space $X \times Y \setminus A \times B$ is path connected in the subspace topology induced by the product topology on $X \times Y$.