1. a) Define the subspace topology.

b) Let $f : X \to Y$ be a continuous map, let $X \times Y$ be the product space (with the product topology) and let

$$\Gamma(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y,$$

equipped with the subspace topology. Show that $\Gamma(f)$ is homeomorphic to X.

2. Let X be a topological space, and let $\sim$ be a relation on X.

a) State what it means for $\sim$ to be an equivalence relation.

b) Define $X/\sim$ and the quotient topology on it.

c) True or false (with proof or counter-example): If X is compact, $X/\sim$ is compact.

d) True or false (with proof or counter-example): If X is normal, $X/\sim$ is normal.

3. Recall that a map $f : X \to Y$ is *open* if $f(U)$ is open whenever $U \subseteq X$ is open. Recall that a *closed map* is one for which $f(A)$ is closed whenever $A \subseteq X$ is closed. Let

$$S^2 := \{v \in \mathbb{R}^3 \mid |v| = 1\}.$$

Give S^2 the subspace topology of the metric topology on $\mathbb{R}^3$. Let $g : S^2 \to S^2$ be continuous and open.

a) Prove that g is a closed map.

b) Prove that g is surjective.

4. a) Define what it means for a map to be nullhomotopic.

b) Prove that if $|\pi_1(X)| < \infty$, then every map $f : X \to T^2$ is nullhomotopic (where $T^2 = S^1 \times S^1$ is the 2-torus).

5. Let

$$SK := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 \leq 1,\ |z| \leq 1\}/\sim \quad \text{and}$$

$$K := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1,\ |z| \leq 1\}/\sim \quad \text{where}$$

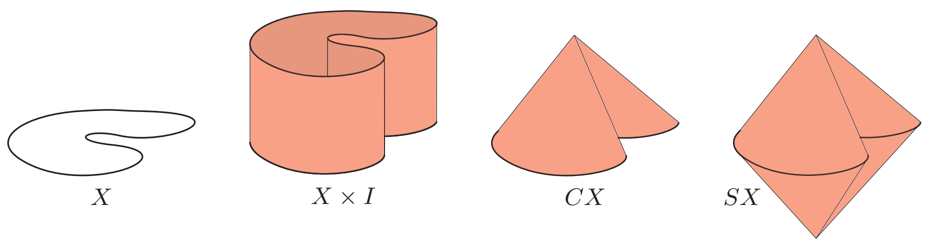
$(x, y, -1) \sim (x, -y, 1)$. Let $[(1, 0, 1)]$ be the base point for each space. Let $\iota : K \to SK$ be the inclusion.

a) Compute $\pi_1(K)$, $\pi_1(SK)$, and the induced map $\iota_* : \pi_1(K) \to \pi_1(SK)$.

b) Construct a non-cyclic abelian group H and a surgective homomorphism $f : \pi_1(K) \to H$.

c) Prove that there is no continuous map $r : SK \to K$ so that $\iota \circ r = \mathrm{id}_K$.

6. Definition: Given a space X, the *cone* CX is the quotient space obtained by collapsing one end of the cylinder $X \times I$. That is $X \times I / \sim$ where $(x, 1) \sim (y, 1)$ for all $x, y \in X$. Furthermore, the *suspension* SX is the quotient space obtained by collapsing both ends end of the cylinder $X \times I$ separately - or equivalently by identifying two copies of CX along $X \times \{0\}$.



Problem: Calculate the cohomology of SX in terms of the cohomology of X (Hint: It is helpful to note that CX is contractible).