**Topology QE I January 2016**

January 23rd, 2016

Last name . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

First name . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

KSU email . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**1.** Consider the finite topological space $X = \{a, b, c, d\}$ with topology

$$\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}.$$

Which of the following topological properties does this space exhibit: compactness, connectedness, path connectedness, separability, contractibility, Hausdorffness, metrizability?

**2.** a) Let $(X, d)$ be a metric space. Show that if $A \subseteq X$ is a compact subset, then it is a closed subset.

b) Give two definitions of connectivity for topological spaces: one using subsets, and one using maps to the discrete space $\{0, 1\}$. Show that both definitions are equivalent.

c) Give an example of a compact and connected space that is not path-connected.

**3.** For each $n \geq 1$, let $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0^2 + \ldots + x_n^2 = 1\}$ be the $n$-th dimensional sphere, and let $\mathbf{b}_{n,0} = (1, 0, \ldots, 0) \in S^n$. Let

$$Y := (\coprod_{i=1}^{n} S^i)/(\mathbf{b}_{j,0} \sim \mathbf{b}_{k,0}) = S^1 \vee S^2 \vee \ldots \vee S^n.$$

a) Find the fundamental group $\pi_1(Y, *)$ where $*$ is an arbitrary base point.

b) Find the homology groups of $Y$.

**4.** Let $S^1 = \{z \in \mathbb{C} \mid \parallel z \parallel = 1\}$. Given $n, m \in \mathbb{Z}$, let $\varphi_n \colon S^1 \to S^1$ be the map defined by $z \mapsto z^n$, and let $\varphi_m \colon S^1 \to S^1$ be the map defined by $z \mapsto z^m$. Finally, let

$$X = S^1 \coprod \{D_1^2, D_2^2\}/\{z \sim \varphi_n(z),\ y \sim \varphi_m(y) \,|\, z \in S^1 = \partial D_1^2,\ y \in S^1 = \partial D_2^2\}.$$

Compute the homology groups of $X$.

**5.** Let $\mathbb{T} = S^1 \times S^1$, the 2-dimensional torus, seen as

$$\mathbb{T} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid \|z_i\| = 1 \, i = 1, 2\}.$$

Let $X = \mathbb{T} \times \mathbb{T}/((z, 0) \sim (0, z))$, with the obvious topology. Briefly justify that $X$ is path-connected, and compute its fundamental group.

**6.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $(\widetilde{X}, \mathcal{T}_{\widetilde{X}})$ and $(\widetilde{Y}, \mathcal{T}_{\widetilde{Y}})$ be covering spaces of $X$ and $Y$ respectively. Show that $\widetilde{X} \times \widetilde{Y}$ is a covering space of $X \times Y$ (with the corresponding product topologies, of course).