Topology Qualifying Exam, June 2016

June 4th, 2016

Last name ....

First name ....

KSU Email ....

Instructions: Use the space below the statement of a problem as well as the next page for the solution. If more space is needed, use the blank pages at the back.

All pages must be submitted. If there is work to be ignored, either cross it out (or otherwise indicate its status) or tape a clean sheet over it to allow the space to be used, being careful not to cover the code at the top.

No references are to be used during the exam.

1. a) Define what it means for a topological space to have each of the following properties: compact, connected, path-connected, contractible.

b) Let X be one of the following topological spaces: the Euclidean space $\mathbb{R}^n$ (for some $n \geq 1$), the Cantor subset of $[0,1]$, the subset $\bigsqcup_{n\geq 1}[\frac{1}{2n^2+1}, \frac{1}{2n^2-1}] \subset \mathbb{R}$, or the torus $S^1 \times S^1$. In each case, briefly justify whether or not X is compact, connected, path-connected, and/or contractible.

2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be nonempty topological spaces. Let $A \subset X$ and $B \subset Y$ be proper subsets. If X and Y are path-connected, show that the complement of $A \times B$ in $X \times Y$ is path-connected.

3. Let $(X, \mathcal{T})$ be a compact space, and let $\{C_j\}_{j \in \mathcal{J}}$ be a family of closed subsets of X. Set $C = \bigcap_{j \in \mathcal{J}} C_j$, and let $U \subset X$ be an open subset containing C. Show that there exists $\{j_1, \ldots, j_n\} \subset \mathcal{J}$ such that

$$C_{j_1} \cap \ldots \cap C_{j_n} \subset U.$$

4. a) Calculate the cohomology of S^n for each $n \geq 0$.

b) Define what it means for a map to be a retraction.

c) Show that there is no retraction of D^{n+1} onto it's boundary S^n for each $n \geq 0$.

5. Consider the space $X_{m,n}$ obtained by attaching two 2-cells to

$$T^2 = \{(z,w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$$

via maps $\varphi_m : S^1 \to T^2$ and $\psi_n : S^1 \to T^2$ defined by $\varphi_m(z) = (z^m, 1)$ and $\psi_n(z) = (1, z^n)$ respectively. That is,

$$X_{m,n} \doteq T^2 \bigsqcup \{D_1^2, D_2^2\} / \{z \sim \varphi_m(z), y \sim \psi_n(y) : z \in \partial D_1^2, y \in \partial D_2^2\}.$$

Calculate the fundamental group of $X_{m,n}$.

6. a) Define what a deck transformation of a covering space is.

b) For the following covering space of $S^1 \vee S^1$, describe the corresponding group of deck transformations.