Topology Qualifying Exam, August 2017

August 26^{th}, 2017

Last name ....

First name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the next page for the solution. If more space is needed, use the blank pages at the back.

All pages must be submitted. If there is work to be ignored, either cross it out (or otherwise indicate its status) or tape a clean sheet over it to allow the space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a passing mark. Credit for completed parts of separate problems may combine to constitute a pass as well. You may use results from one part of a problem (even if you did not solve it) in your solution to a subsequent part.

No references are to be used during the exam.

1. (10 pts) Consider the finite topological space $X = \{0, a, b, c\}$ with topology induced by the subbasis $\{\{a, 0, b\}, \{a, 0, c\}, \{b, 0, c\}\}$.

a) List all of the open sets in the topology on X.

b) Prove in detail that X is path connected.

c) Which of the following topological properties does X exhibit – compactness, T_0-ness, T_1-ness, Hausdorffness, separability, contractibility?

2. (10 pts) The word "embedding" is used differently in point-set topology and in differential topology. Give both definitions, and prove that a smooth function between smooth manifolds which is an embedding in the sense of differential topology, when regarded as a continuous function between the underlying topological manifolds is an embedding in the sense of point-set topology.

3. (10 pts) The usual definition of the real projective plane, $\mathbb{RP}^2$, is the quotient space of $\mathbb{R}^3 \setminus \{\vec{0}\}$ by the equivalence relation $\vec{x} \equiv \vec{y}$ if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\vec{x} = \lambda \vec{y}$.

It can also be described as a quotient of the disjoint union of an open disk and an open Möbius strip by an equivalence relation that identifies an open annulus along the boundary of the disk with the annulus obtained from the Möbius strip by removing a circle which generates the fundamental group of the Möbius strip. Use the second description – you do **not** need to prove it is equivalent to the first – to find

a) The fundamental group $\pi_1(\mathbb{RP}^2)$ using the Seifert-vanKampen Theorem.

b) The deRham cohomology of $\mathbb{RP}^2$ by using an appropriate Meyer-Vietoris sequence.

4. (10 pts) Consider the differential 3-form

$$\omega = w^2\, dx \wedge dy \wedge dz + xw\, dw \wedge dy \wedge dz + yw\, dw \wedge dx \wedge dz$$

defined on $\mathbb{R}^4$.

a) Find $\int_{S^3} \omega$, where S^3 is the standard unit sphere $\{(w,x,y,z) \mid w^2 + y^2 + z^2 + w^2 = 1\}$.

b) Is there a differential 2-form η such that $\omega = d\eta$? Justify your answer.

5. (10 pts)

a) Define what it means to be a covering space and what it means for a map to be nullhomotopic.

b) Prove that if $|\pi_1(X)| < \infty$, then every map $f : X \to T^n$ is nullhomotopic (where $T^n = S^1 \times \cdots \times S^1$ is the n-torus).

6. (10 pts)

a) Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is f caries closed sets to closed sets).

b) Give an example of a continuous map $f : X \to Y$, where either X is noncompact, Y is not Hausdorff, or both, such that f fails to be closed.