TOPOLOGY QUALIFYING EXAM, JUNE 2017

– JUNE 5TH, 2017 –

Last name ....

First name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the next page for the solution. If more space is needed, use the blank pages at the back.

All pages must be submitted. If there is work to be ignored, either cross it out (or otherwise indicate its status) or tape a clean sheet over it to allow the space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all six problems. Four complete solutions will earn a passing mark. Credit for completed parts of separate problems may combine to constitute a pass as well.

No references are to be used during the exam.

1. (10 pts)

(a) Define what it means to be a covering space and what a lifting of a map is with respect to a covering map.

(b) Show that given $p : \widetilde{X} \to X$ a covering map, every map $f : S^n \to X$ lifts to $\widetilde{X}$ provided $n > 1$.

(c) Give an example where the above statement fails when $n = 1$.

2. (10 pts) A $2n$-dimensional smooth manifold X is *symplectic* if it admits a closed 2-
$\omega \in \Omega^2(X)$ such that $\omega^n \doteq \omega \wedge \omega \wedge \ldots \wedge \omega \in \Omega^{2n}(X)$ is a volume form; in particular,
may assume that

$$\int_X \omega^n > 0.$$

(a) Show that ω^k is closed for each $k \in \{1, \ldots, n\}$.

(b) Show that ω^k cannot be exact for each $k \in \{1, \ldots, n\}$.

(c) What does this say about $H^{2k}_{\mathrm{DR}}(X)$ for any symplectic manifold X?

3. (10 pts)

(a) Define what it means for a map between smooth manifolds to be a submersion.

(b) Use the inverse function theorem to prove:
If $f : X \to Y$ is a submersion, then for every $x \in X$, there exist charts $(U, h_U) \ni x$ and $(V, h_V) \ni f(x)$ on which $h_V(f(h_U^{-1})$ is given by

$$(x_1, \dots, x_{dim(Y)}, \dots, x_{dim(X)}) \mapsto (x_1, \dots, x_{dim(Y)}).$$

4. (10 pts)

(a) Define what is means to be a metric space.

(b) Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points $x_1, x_2, \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an embedding (i.e. that f is a homeomorphism onto its image $f(X)$).

5. (10 pts) Recall that a map $f : X \to Y$ is *open* if $f(U)$ is open whenever $U \subset X$ is.

(a) Define what it means to be a quotient map.

(b) Suppose that $p : X \to X^*$ and $q : Y \to Y^*$ are each *open* quotient maps. Show that $p \times q : X \times Y \to X^* \times Y^*$ is also a quotient map.

6. (10 pts) Consider the space $X_{m,n,k}$ obtained by attaching two 2-cells to

$$T^2 = \{(z,w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$$

via maps $\varphi_{m,N} : S^1 \to T^2$ and $\psi_k : S^1 \to T^2$ defined by $\varphi_{m,n}(z) = (z^m, \overline{z}^n)$ and $\psi_k(z) = (1, z^k)$ respectively. That is,

$$X_{m,n,k} \doteq T^2 \bigsqcup \{D_1^2, D_2^2\}/\{z \sim \varphi_{m,n}(z), y \sim \psi_k(y) : z \in \partial D_1^2, y \in \partial D_2^2\}.$$

Calculate the fundamental group of $X_{m,n,k}$.