Topology QE I Exam
Spring 2018

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all six problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

a) Consider the space $X = \{0\} \cup \{\frac{1}{2^n} \mid n = 1, 2, 3, \ldots\}$ in the subspace topology induced by the usual metric (or equivalently order) topology on $\mathbb{R}$.

b) Prove in detail that X is compact.

c) Which of the following topological properties does X exhibit – Hausdorffness, separability, second countability, connectedness, contractibility? For each explain *briefly* why it does or does not exhibit the property.

2. (10 pts)

Let $H^k(X)$ denote the k-th deRham cohomology group of a manifold (with or without boundary) X.

Prove in detail that $H^k(X \coprod Y) \cong H^k(X) \times H^k(Y)$, where $\coprod$ denotes the disjoint union of manifolds.

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3. (10 pts)

Give a presentation for the fundamental group of the space obtained by removing a disk from the torus $S^1 \times S^1$ and a disk from the real projective plane $\mathbb{R}P^2$, and gluing the bounding circles by a map of degree 1.

4. (10 pts)

a) Define what it means for a space to be connected.

b) Define what it means for a space to be path connected.

c) Prove that if a space X is path connected then X is connected.

d) Give a counter-example to the converse implication.

5. (10 pts)

a) Prove the "Stack of Records Theorem":

If $y \in Y$ is a regular value of a smooth map $f : X \to Y$ with X compact and $\dim(X) = \dim(Y)$, then $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_n\}$ and moreover there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union of opens $V_1, \ldots, V_n$ in X with $x_i \in V_i$ and $f|_{V_i} : V_i \to U$ a diffeomorphism, for $i = 1, \ldots, n$.

b) Give a counterexample to the same assertion with the compactness of X omitted, in which, not only is the preimage of y infinite, but there is no neighborhood U of y whose preimage is a disjoint union of open sets each mapped diffeomorphically to U by f.

6. (10 pts)

Suppose that $X = \partial W$ where W is a compact manifold of dimension $k+1$. Let $f : X \to Y$ be a smooth map. Let ω be a closed k-form on Y. Prove that if f extends to all of W, then $\int_X f^*\omega = 0$.