Topology QE I Exam
August 2019

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

Consider the set $\mathbb{Z}$ of integer numbers endowed with the cofinite topology (only finite sets and the entire set are closed).

(a) Is this space compact? Prove your statement.

(b) Is it connected? Prove.

(c) Is it T_0? T_1? Hausdorff? regular? normal? metrizable? Explain.

2. (10 pts) Let $X = S^1 \vee S^2 \vee S^3$.

(a) Compute $\pi_1(X)$.

(b) Describe the universal covering of X.

3. (10 pts)

Let $\Sigma = \mathbb{R}P^2 \# \mathbb{R}P^2$.

(a) Compute the Euler characteristic of Σ.

(b) Does Σ admit a 2-sheeted covering by itself? Describe such covering if your answer is yes, otherwise prove that it is not possible.

4. (10 pts)

Let $g : N \to \mathbb{R}$ be a C^∞ function on the manifold N. Prove that a nonempty regular level set $S = g^{-1}(c)$ is a smooth proper submanifold of N of codimension 1.

5. **(10 pts)** Recall that the exterior derivative d on a manifold M is defined as an $\mathbb{R}$-linear map $\Omega^*M \to \Omega^*M$ such that $d \circ d = 0$, the form $d(f)$ is the differential df of any smooth function f on M, and for any pair of differential forms $a, b \in \Omega^*M$ one has

$$d(a \wedge b) = d(a) \wedge b + (-1)^k a \cdot d(b),$$

where k is the degree of the form a. Using these properties of d prove that if two differential forms w and w' agree on a neighborhood of a point $p \in M$, then $d(w)|_p = d(w')|_p$.

6. (10 pts) The antipodal map $a : S^n \to S^n$ is the map $x \to -x$. Show that the antipodal map is orientation preserving if and only if n is odd. Deduce that $\mathbb{R}P^{2n+1}$ is orientable for each $n \geq 0$.