Topology Qualifying Examination
August 2020

Appropriate instructions for a Zoom-proctored exam distributed how?

1. (a) Define what it means for a topological space to be each of the following: discrete, Hausdorff, T_1.
   (b) A space is said *totally disconnected* if its only connected subsets are singletons. Prove that for finite topological spaces each property of part (a) is equivalent to total disconnectedness.
   (c) Give an example of a (necessarily infinite) Hausdorff space which is totally disconnected, but not discrete.

2. (a) Define what it means for a topological space to be compact.
   (b) Prove that any subspace of a Hausdorff space which is compact in the subspace topology is closed as as subset of the ambient space.

3. Prove that if $A \subset X$ is a deformation retract of X, then for any space Y there is a bijection between the set of homotopy classes of continuous maps from Y to X and the set of homotopy classes of continuous maps from Y to A.

4. (a) Determine (with a proof) the fundamental group $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2)$ of the wedge sum of two projective planes.
   (b) Prove that $\mathbb{R}P^2 \vee \mathbb{R}P^2$ does not admit a 5-sheeted connected normal covering. (Hint: use the fact that the only group of order five is the cyclic one $\mathbb{Z}_5$.)

5. Let $p \subset \mathbb{R}^3$ be the plane $z = 1$, and let $\pi \colon \mathbb{R}^3 \setminus p \to \mathbb{R}^2$ be the stereographic projection on the (x, y)-coordinate plane from the point $(0, 0, 1)$.
   (a) Let $\omega = dx \wedge dy \in \Omega^2(\mathbb{R}^2)$. Compute $\pi^*\omega \in \Omega^2(\mathbb{R}^3 \setminus p)$ in the natural (x, y, z) coordinates on $\mathbb{R}^3 \setminus p$.
   (b) Is $\pi^*\omega$ closed? Is it exact?

6. Let $\ell \subset \mathbb{R}^3$ be a straight line and $A \in \mathbb{R}^3 \setminus \ell$ be a point in its complement. Compute the de Rham cohomology $H^*(\mathbb{R}^3 \setminus (\ell \cup \{A\}))$.