Topology Qualifying Examination
June 2020

Instructions: This is a three-hour, closed-book exam. No texts, notes, or aid from other people or resources on the internet, are allowed.

1. (a) Some authors call a subset of a topological space X a "clopen" if it is both open and closed. Prove that there is a bijection between the set of clopens of a space X and the set of continuous functions from X to the two-point discrete space.

   (b) Define what it means for a space to be connected.

   (c) Use your result in part (a) to prove that a space X is connected if and only if there are exactly two continuous functions from X to the two-point discrete space.

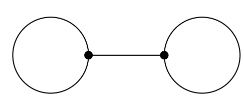
2. (a) Define what it means for a topological space to be T_1 and what it means to be normal.

   (b) Prove that a compact subspace of a normal T_1 space is normal and T_1 (in the subspace topology).

3. (a) Using the fact that $\pi_1(S^1, *) \cong \mathbb{Z}$, the Seifert-VanKampen Theorem, and the fact that fundamental groups are invariant under deformation retracts as starting points, give a fully justified calculation showing that the fundamental group of a bouquet of n circles, $\vee_{i=1}^{n}(S^1, *)$ is the free group on n-generators.

   (b) Describe the universal covering space of a bouquet of two circles, $(S^1, *) \vee (S^1, *)$. (Hint: it is the geometric realization of an infinite graph.)

4. Let X be the following one-dimensional CW complex:

   (a) Prove that X is homotopy equivalent to $S^1 \vee S^1$.

   (b) Give an example of a connected 3-sheeted non-normal covering of $S^1 \vee S^1$.

   (c) Give an example of a connected 3-sheeted non-normal covering of X.

   (Make sure that your pictures indicate clearly both the covering space and the covering map.)

5. Prove that a product $M \times N$ of two non-empty smooth maniflods M and N is orientable if and only if each factor is orientable.

6. Let ℓ be a straight line in $\mathbb{R}^3$. Compute the de Rham cohomology of $\mathbb{R}^3 \setminus \ell$.