Topology Qualifying Examination
August 2021

1. Let X be a totally ordered set. Give X the topology generated by the subbasis consisting of all sets of the form:

$$(a, \infty) = \{x \in X : x > a\}$$
$$(-\infty, a) = \{x \in X : x < a\}.$$

   (a) Show that each set of the form (a, b) is open in X and each set of the form $[a, b]$ is closed. (Here (a, b) and $[a, b]$ are defined as in $\mathbb{R}$)

   (b) Show that X is Hausdorff.

   (c) For any two points $a, b \in X$, with $a < b$, prove that $\overline{(a, b)} \subseteq [a, b]$. Give an example that proves the equality doesn't always hold.

2. (a) Prove that if X is compact and non-empty, Y is connected and Hausdorff, and $f : X \to Y$ is a continuous open map, then f is surjective.

   (b) Give an example to show the previous statement does not necessarily hold when X is not compact.

3. Let $X = S^1 \vee S^2$.

   (a) Compute the fundamental group of X.

   (b) Describe the universal covering of X.

4. (a) Give the intersection theoretic definition of Euler characteristic applicable to smooth manifolds.

   (b) Prove, using this definition and the homotopy invariance of intersection numbers, that if M is a smooth manifold, then $\chi(M \times S^1) = 0$.

5. (a) State the theorem called Stokes Theorem applicable to differential forms and generalizing the classical theorem of the same name.

   (b) Let $\Sigma = \{(x, y, z) | x^2 + y^2 + z^2 = 4\}$ and $H = \Sigma \cap \{(x, y, z) | x \geq 0\}$. Use Stokes Theorem to evaluate the following integrals of differential 2-forms:

      i.
$$\int_\Sigma x dy \wedge dz$$

      ii.
$$\int_\Sigma 2xy dx \wedge dz + x^2 dy \wedge dz$$

      iii.
$$\int_H 2xy dx \wedge dz + x^2 dy \wedge dz$$

6. (a) Use a Meyer-Vietoris sequence and homotopy invariance of deRham cohomology to prove

   Theorem For any manifold X, $H^n_{dR}(X \times S^1) \cong H^n_{dR}(X) \oplus H^{n-1}_{dR}(X)$, where $H^{-1}_{dR}(X)$ is understood to be 0.

   (b) Use the preceding theorem and induction to compute the deRham cohomology of the k-dimensional torus (that is the product of k copies of S^1).