Topology Qualifying Examination
June 2021

1. Let X be a set and p an arbitrary point in X.

   (a) Show that
   $$\mathcal{T} = \{U \subseteq X \mid U = \emptyset \text{ or } p \in U\}$$
   is a topology on X. This is called the particular point topology.

   (b) Show that $(X, \mathcal{T})$ is a connected topological space.

   (c) When is $(X, \mathcal{T})$ compact? Justify your answer.

2. Let $\mathbb{T}^n = S^1 \times \cdots \times S^1$ be the n-dimensional torus.

   (a) What is the universal covering space of $\mathbb{T}^n$.

   (b) Assume $n \geq 2$ and let $f : S^n \to \mathbb{T}^n$ be a continuous maps. Show that f is null-homotopic.

3. (a) State the Seifert-Van Kampen theorem.

   (b) Let K be the space obtained from from a square $[0,1] \times [0,1]$ by identifying opposite sides as follows: $(x, 0) \sim (x, 1)$ for all x, and $(0, y) \sim (1, 1-y)$ for all y. Compute $\pi_1(K)$ using the Seifert-Van Kampen theorem.

4. Recall that a property P of smooth maps is said to be stable if whenever X is a compact manifold, Y a manifold, and $h : X \times [0,1] \to Y$ is a smoooth homotopy, if $h(-,0) : X \to Y$ satisfies P, then there is an $\epsilon > 0$ such that $t < \epsilon$ impiles $h(-,t) : X \to Y$ satisfies P.

   (a) Without proof, which of the following properties are stable: local diffeomorphism, surjectivity, injectivity, transversality to a fixed submanifold Z of the target Y?

   (b) If any of the listed properties are not stable, give an example of a compact manifold and homotopy of maps from it which begins with a map exhibiting the property, but for which no $\epsilon > 0$ as required in the definition exists.

   (c) If any of the listed properties are stable, give an example of a non-compact manifold and a homotopy of maps from it which begins with a map exhibiting the property, but for which no $\epsilon > 0$ as required in the definition exists.

5. Let $j : S^1 \times S^1 \to \mathbb{R}^4$ be the inclusion give in terms of angular coordinates θ and ϕ on the first and second factors, respectively, by
   $$(\theta, \phi) \mapsto (\cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi)),$$
   and consider the differential 2-form

$$\omega := xdy \wedge dz + ydx \wedge dz + ydz \wedge dt + zdy \wedge dt$$

on $\mathbb{R}^4$.

Find

$$\int_{S^1 \times S^1} j^*\omega.$$

6. Without invoking the Künneth formula (though if you know it, it will tell you what the answer will be), but using only the well-known deRham cohomology groups of the point and of spheres, and the homotopy invariance of deRham cohomology and Mayer-Vietoris Theorem, compute the deRham cohomology of $S^k \times S^n$ for $k, n \geq 1$. (Hint: the case $k = n$ will be a bit different.)