Topology Qual, August 2022

1. Is it true that any continuous map from the real line to the Cantor set must be constant? Either prove your answer or provide a counter-example.

2. Give the definition of a normal space. A partition of unity subordinate to an open cover $\{U_\alpha\}$ is, in particular, a collection of continuous functions $\varphi_\alpha : X \to [0,1]$ with $\varphi_\alpha^{-1}((0,1]) \subset U_\alpha$. Show that a Hausdorff space such that every open cover has a subordinate partition of unity is normal.

3. The 3-dimensional torus is defined as a product of three circles $T^3 = S^1 \times S^1 \times S^1$.

   (a) Compute $\pi_1(T^3)$.

   (b) Compute $\pi_1(T^3 \setminus \text{point})$ (hint: use (a) and the Van Kampen theorem).

4. Consider the unit sphere $i\colon S^2 \subset \mathbb{R}^3$ and the restriction $i^*\omega$ of the form $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$.

   (a) Give an example of a smooth atlas $\mathcal{A}$ on S^2.

   (b) Compute $i^*\omega$ in two non-empty charts of $\mathcal{A}$.

5. Give two examples of infinite sheeted coverings of the wedge of two circles, such that one is regular, while the other one is not.

6. Define the de Rham cohomology of a smooth manifold M. Prove that the wedge product induces a well-defined map $H^p(M) \times H^q(M) \to H^{p+q}(M)$.