Topology Qualifying Exam, June 2022

INSTRUCTIONS

This is a closed-book exam. No written material, electronic tools, or communication with others is permitted. You have three hours to work and 20 minutes to upload solutions.

Four questions correctly solved (up to minor errors) will earn a pass on this exam. Parts of questions may in some cases combine to count as one full problem.

Do not write your name or any other identifying information on your work.

Submitting your work:

- Find the email from Crowdmark Mailer with subject “Graduate Program 2021-2022 - New Assignment: Topology" in the subject line. Follow steps given.
- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to sarahrez@ksu.edu.
- Before leaving the library, give all hard copies to the proctor.

Topology Qual Ideas for Spring 2022

1. Let $X = \mathbb{Z}$ and $Y = \mathbb{R}^2$ with the standard topologies. Is X^2 homeomorphic to X? Is Y^2 homeomorphic to Y? Prove your answers.

2. Consider the map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, z) \mapsto (y^2 + z^2, x^2 + z^2, x^2 + y^2)$.

   (a) Prove that F induces a well-defined map $[F]\colon \mathbb{R}P^2 \to \mathbb{R}P^2$ of the sets of straight lines passing through the origin.

   (b) Prove that $[F]$ is smooth (where $\mathbb{R}P^2$ is taken with its standard smooth structure).

   (c) Find all critical points and all critical values of $[F]$.

3. Let $S = \{(x, y, z) \mid x^2 + y^2 + 4z^2 = 4,\ z \geq 0\}$. Orient S with $dy \wedge dx$ at $(0, 0, 1)$. Compute
$$\int_S 4x^3 dx \wedge dz + dx \wedge dy.$$

4. Define what it means for a space to be path connected. Define the quotient topology. A space is *arc connected* if any pair of distinct points may be realized as the endpoints of a topologically embedded path. Consider $\mathbb{R} \times \{\pm 1\}$ with the standard topology inherited as a subspace of the Euclidean plane. Let $X = (\mathbb{R} \times \{\pm 1\}) / \sim$ where $(x, +1) \sim (x, -1)$ for $x \neq 0$ and give X the quotient topology. Prove that X is path connected but not arc connected.

5. Let $i : S^1 \to D^2$ be the boundary inclusion. Let $j : S^1 := \{z \in \mathbb{C} \mid |z| = 1\} \to S^1 \times S^1$ be given by $j(z) = (z^2, 1)$. Compute the fundamental group of $(S^1 \times S^1) \cup_{j, S^1, i} D^2$.

6. Consider a 3-fold cover $\pi\colon \#_g T^2 \to \#_2 T^2$.

   (a) Draw a picture of such a cover.

   (b) Find the genus g.

   (c) Prove that the genus g of the total space is the same for any cover.

   (d) For extra credit, draw two such covers, one regular and one irregular.