Topology QE I Exam
August 2023

Name . . . . . .

KSU Email . . . . . .

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. **(10 pts)** Define what it means for two maps $f, g : X \to Y$ (between two smooth manifolds X and Y) to be smoothly homotopic. If $f, g : X \to Y$ are smoothly homotopic where X is a closed k-manifold and ω is a closed k-form on Y, prove that $\int_X f^*\omega = \int_X g^*\omega$.

2. **(10 pts)** Consider the set $\mathbb{Z}$ with the topology $\mathcal{T} = \{\mathbb{Z}, \emptyset\} \cup \{[k, +\infty) \cap \mathbb{Z} \,|\, k \in \mathbb{Z}\}$.

   (a) Is $(\mathbb{Z}, \mathcal{T})$ path-connected?

   (b) Which of the separations axioms T_0, T_1, T_2 does it satisfy?

3. **(10 pts)** True or false and why?

   (a) Any connected closed surface with even Euler characteristic is orientable.

   (b) Any connected closed surface with strictly positive Euler characteristic has abelian fundamental group.

4. **(10 pts)**

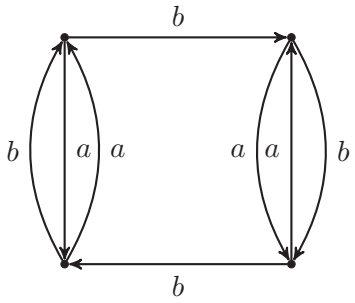
   (a) State Sard's theorem.

   (b) Let M be an n-dimensional smooth manifold. Show that there is no smooth surjective map $f : M \to S^{n+1}$.

   (c) Prove that any smooth map $f : M \to S^{n+1}$ is homotopic to a constant.

5. **(10 pts)**

(a) Is the following cover of the wedge of two circles $S^1_a \vee S^1_b$ regular?



(b) Determine its group of covering transformations.

6. **(10 pts)** Consider the first uncountable ordinal Ω with its induced order topology. Prove that Ω is not compact but is sequentially compact.