Topology Qual for June 2023

1. (a) State the T_1 axiom.

   (b) It is known that $[0, 1]$ (with its standard topology) cannot be obtained as a countable disjoint union of closed non-empty subsets. Use this fact to show that any countable topological space satisfying T_1 is not path-connected.

2. (a) Give the definition of a smooth atlas.

   (b) Give the definition of a smooth structure.

   (c) Let $p : E \to M$ be an at-most-countably-sheeted covering map with M a smooth manifold. One can show that E must be a topological manifold. Define a natural smooth structure on E based on these data and prove that it is a smooth structure.

3. Compute the de Rham cohomology $H^*(\mathbb{R}P^2 \setminus \{\text{two points}\})$. (You can use as given the de Rham cohomology of a point and that of a circle.)

4. Consider the set $\mathbb{Q}$ of rationals with its usual topology. Define a quotient relation $a \sim b$ if and only if $a = b$ or $a, b \in \mathbb{Z}$.

   (a) Prove that $X := \mathbb{Q}/\sim$ is not first countable. (It is an example of a countable but not first countable space.)

   (b) Is X second second-countable? Is it metrizable?

5. Recall that the Lie bracket $[X, Y]$ of two vector fields X and Y (viewed as derivations) is given by $[X, Y](f) = X(Y(f)) - Y(X(f))$. Define the Lie derivative of a 1-form α with respect to a vector field X by

$$(L_X\alpha)(Y) = X(\alpha(\overline{Y})) - \alpha([X, \overline{Y}]).$$

   Here one takes evaluation of $L_X\alpha$ on a tangent vector Y at some point, and $\overline{Y}$ is any vector field extending Y. Note that the Lie derivative of a 1-form is a 1-form. Assuming as given that the formula is well-defined and does not depend on the choice of $\overline{Y}$, compute $L_{x\partial_x}(x^2\,dx + xy\,dy)$.

6. (a) Give a rigorous definition of a Klein bottle. Hint: consider a quotient of a square.

   (b) Compute a presentation for the fundamental group of the Klein bottle.

   (c) Define a covering projection from $\mathbb{R}^2$ to the Klein bottle.

   (d) Prove that the fundamental group of the Klein bottle is not abelian (preferably using the covering from (c)).