Topology/Geometry QE1

05/28/2024

Name: ______________________________

KSU Email: ______________________________

Instructions: You have three hours to do your work and fifteen minutes to upload the solutions.

0) Four full problems correctly done will earn a pass. Parts of several problems may combine to count for a full problem.

1) Do not write your name on any of the pages.

2) This is a closed book exam: no books, no notes, no calculators etc. Only plain papers and pens should be on your table.

3) You must have your camera on showing your hands and, if possible, at least part of your faces during the exam.

4) After you receive the exam online by email, if you want you can print it or you can keep it open on your laptop or cellphone. After that you are allowed to use your computer or any electronic device during the exam only to read the exam and later to upload it on Crowdmark. Also you can communicate to the examiner via private chat in Zoom.

5) In case you lose internet connection at some point, you can continue your exam, however the examiners might consider to have an oral reexamination with you where you would need to explain steps in your work. You can be asked additional questions. In case the loss of connection is long, you can contact the examiner.

6) The exam is supposed to take 3 hours not counting the time of printing or accessing the problems and uploading your test. If you want you can have a bit of extra time, but the exam must be uploaded to Crowdmark no less than 10 minutes after the exam

Submitting your work:

- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to tinaande@ksu.edu.

1. Let X be a finite topological space
   (a) Prove that X is separable
   (b) Prove that X is Hausdorff if and only if it is discrete.

2. (a) Define what it means for a space to be path-connected and what it means to be compact.

   (b) Consider the space D_{FR}, the closed unit disk $\{\vec{x} \in \mathbb{R}^2 | \, |\vec{x}| \leq 1\}$ with the topology induced by the "French railway metric"

$$d(\vec{x}, \vec{y}) = \begin{cases} |\vec{x} - \vec{y}| & \text{if } \vec{x}, \vec{y} \text{ and } \vec{0} \text{ are colinear,} \\ |\vec{x}| + |\vec{y}| & \text{otherwise.} \end{cases}$$

   Prove that D_{FR} is path-connected, but is not compact.

3. (a) Describe the universal covering space of $\mathbb{R}P^2 \vee S^1$, including a description of the covering map.

   (b) What is $\pi_1(\mathbb{R}P^2 \vee S^1)$? Justify your answer.

4. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map $F(x, y) = (e^x \cos y, e^x \sin y)$.

   (a) Compute the induced map of tangent bundles $F_*\colon T\mathbb{R}^2 \to T\mathbb{R}^2$ in coordinates (x, y, X, Y) on $T\mathbb{R}^2 = \mathbb{R}^4$, where a point with coordinates (x, y, X, Y) is the vector $X\partial_x + Y\partial_y \in T_{(x,y)}\mathbb{R}^2$.

   (b) Is F_* a local diffeomorphism? Justify your answer. (Note that the question is about F_* and not about F.)

5. Let $F = (f_1, f_2, \ldots, f_n)\colon M^m \to \mathbb{R}^n$ be a smooth map. Prove that $x \in M$ is a critical point of F if and only if $(df_1)_x, \ldots, (df_n)_x \in T_x^*M$ are linearly dependent if and only if $(df_1 \wedge \ldots \wedge df_n)_x = 0$.

6. Let $id, r, c\colon S^2 \to S^2$ be the identity $id\colon (x, y, z) \mapsto (x, y, z)$, reflection $r\colon (x, y, z) \mapsto (x, y, -z)$, and constant $c\colon (x, y, z) \mapsto (1, 0, 0)$ maps, respectively. Compute the induced maps $id^*, r^*, c^*\colon H^*(S^2) \to H^*(S^2)$ in de Rham cohomology. Use your answer to prove that none of the maps is smoothly homotopic to any other.