Sample Topology Qualifying Exam

1. Let $\pi : \widetilde{X} \longrightarrow X$ be a covering map. Suppose that $f,\ g : Y \longrightarrow \widetilde{X}$ are continuous maps such that $\pi \circ f$ and $\pi \circ g$ are equal and assume that f and g agree at $y_0 \in Y$. Show that if Y is connected, then $f = g$.

2. Let $R = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ and consider the quotient space $X = R/\sim$ where $z \sim e^{2\pi i/3}z$ for $|z| = 1$ and $z \sim e^{2\pi i/5}z$ for $|z| = 2$. Thus X is obtained from the annulus by identifying certain points on its two boundary circles. Describe the fundamental group $\pi_1(X, *)$. (Hint: You may cut R along the circle of radius $3/2$ and apply the van Kampen theorem.)

3. (a) Give the interesection-theoretic definition of Euler characteristic applicable to compact smooth manifolds.

   (b) State and prove a general theorem about the Euler characteristic of manifolds for which there exists a fixed-point-free self-map homotopic to the identity map.

   (c) For which of the following manifolds does the theorem of part (b) allow one to compute the Euler characteristic: S^2, $S^1 \times S^1 \times S^1$, S^3?

4. Prove that if Z_0 and Z_1 are compact oriented p-dimensional submanifolds, such that they are cobordant in X, that is, there is a compact oriented $p+1$-dimensional submanifold with boundary of X, whose boundary is $Z_0 \coprod -Z_1$ and ω is a closed p-form on X, then

$$\int_{Z_0} \omega = \int_{Z_1} \omega.$$

5. Consider the finite topological space with underlying set $\{a, b, c, d\}$ and topology $\{\emptyset, \{a\}, \{c\}, \{a, b, c\}, \{c, d, a\}, \{a, b, c, d\}\}$. Which of the following topological properties does it satisfy: connected, path connected, compact, Hausdorff, T_0, T_1, simply connected? For each property briefly prove your assertion that the space does or does not satisfy it.

6. Let X be the subspace of $\mathbb{R}^2$ consisting of the union of the two sets $A = \{(0, y) \mid y \in [-1, 1]\}$ and $B = \{(x, \sin(1/x)) \mid x \in (0, \pi/2]\}$.

   (a) Show that X is a compact space.

   (b) Prove that X is connected but not path-connected.