Nagy Trig

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We reorganize and consolidate the information in Nagy's textbook: https://www.math.ksu.edu/~nagy/math150/trig-book.html

§1.2. Right Triangle Trigonometry

The Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$
$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Just remember that these pairs are reciprocal:

- sin and csc
- cos and sec
- tan and cot

The Ratio Identities

$$\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{\cos\theta}{\cot\theta} \qquad \csc\theta = \frac{\cot\theta}{\cos\theta} = \frac{\sec\theta}{\tan\theta}$$
$$\cos\theta = \frac{\cot\theta}{\csc\theta} = \frac{\sin\theta}{\tan\theta} \qquad \sec\theta = \frac{\tan\theta}{\sin\theta} = \frac{\csc\theta}{\cot\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sec\theta}{\csc\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\csc\theta}{\sec\theta}$$

Just remember that $\tan = \frac{\sin}{\cos}$. Everything else can be derived.

The Product Identities Unnecessary. Just rearrange / manipulate the ratio identities.

The Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Just have to remember the first equation. The other two can be derived by dividing by $\cos^2$ and $\sin^2$, respectively.

§2.3. Trigonometric Identities

Supplement Formulas

$$\sin(\pi-\alpha) = \sin\alpha; \qquad \csc(\pi-\alpha) = \csc\alpha;$$
$$\cos(\pi-\alpha) = -\cos\alpha; \qquad \sec(\pi-\alpha) = -\sec\alpha;$$
$$\tan(\pi-\alpha) = -\tan\alpha; \qquad \cot(\pi-\alpha) = -\cot\alpha.$$

The transformation $\alpha \mapsto \pi - \alpha$ is a horizontal reflection. $(x,y) \mapsto (-x,y)$

Add π Formulas

The transformation $\alpha \mapsto \pi + \alpha$ is a 180° rotation (or two reflections). $(x, y) \mapsto (-x, -y)$

$$\sin(\pi + \alpha) = -\sin\alpha; \qquad \csc(\pi + \alpha) = -\csc\alpha;$$
$$\cos(\pi + \alpha) = -\cos\alpha; \qquad \sec(\pi + \alpha) = -\sec\alpha;$$
$$\tan(\pi + \alpha) = \tan\alpha; \qquad \cot(\pi + \alpha) = \cot\alpha.$$

Formulas for Negatives

The transformation $\alpha \mapsto -\alpha$ is a vertical reflection. $(x, y) \mapsto (x, -y)$

Alternatively, sine is an odd function, cosine is an even function.

$$\sin(-\alpha) = -\sin\alpha; \qquad \csc(-\alpha) = -\csc\alpha;$$
$$\cos(-\alpha) = \cos\alpha; \qquad \sec(-\alpha) = \sec\alpha;$$
$$\tan(-\alpha) = -\tan\alpha; \qquad \cot(-\alpha) = -\cot\alpha.$$

Post Midterm 2 material

Complement Formulas

Property of co-functions

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha; \qquad \csc\left(\frac{\pi}{2} - \alpha\right) = \sec\alpha;$$
$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha; \qquad \sec\left(\frac{\pi}{2} - \alpha\right) = \csc\alpha;$$
$$\tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha; \qquad \cot\left(\frac{\pi}{2} - \alpha\right) = \tan\alpha.$$

Anti-Complement Formulas

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha; \qquad \csc\left(\frac{\pi}{2} + \alpha\right) = \sec\alpha;$$
$$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha; \qquad \sec\left(\frac{\pi}{2} + \alpha\right) = -\csc\alpha;$$
$$\tan\left(\frac{\pi}{2} + \alpha\right) = -\cot\alpha; \qquad \cot\left(\frac{\pi}{2} + \alpha\right) = -\tan\alpha.$$

Negative Complement Formulas

$$\sin\left(\alpha - \frac{\pi}{2}\right) = -\cos\alpha; \qquad \csc\left(\alpha - \frac{\pi}{2}\right) = -\sec\alpha;$$
$$\cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha; \qquad \sec\left(\alpha - \frac{\pi}{2}\right) = \csc\alpha;$$
$$\tan\left(\alpha - \frac{\pi}{2}\right) = -\cot\alpha; \qquad \cot\left(\alpha - \frac{\pi}{2}\right) = -\tan\alpha.$$

§2.5. General Sinusoidal Functions and Their Graphs

The *general form* of a sinusoidal function is

$$A\sin(Bx + C) + D$$

or

$$A\cos(Bx + C) + D.$$

For this class, we assume $D = 0$.

- A corresponds to Amplitude
- B corresponds to Frequency (and corresponds inversely to wavelength / Period)
- C corresponds to Phase Shift

Convention: I will use capital A, B, C for the general form and lowercase a, b, c for the "standard form" (see below). These variables have the same conceptual meanings, but generally may change when converting from general form to standard form.

The *"standard form"* for a sinusoidal function is

$$a \sin(bx + c)$$

where a and b are both positive.

This form is preferred for sketching, as it simplifies what we need to consider

General form $\rightsquigarrow$ *Standard form*

This distills/generalizes Example 3 from Lecture 2022-02-23

- If the function uses cos(): Convert it with the anti-complement formula

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right).$$

  If the resulting function is not already in standard form, use one of the three cases below.

- Case (both A and B are negative): Use the formula for negatives

$$-\sin(\theta) = \sin(-\theta)$$

  to convert the function to standard form.

- Case (Just A is negative): Use the "add π" formula

$$-\sin\theta = \sin(\theta + \pi)$$

  to convert the function to standard form.

- Case (Just B is negative): Use the supplement formula

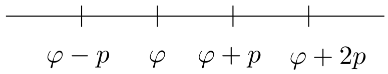
$$\sin\theta = \sin(\pi - \theta)$$

  to convert the function to standard form.

Standard form $\rightsquigarrow$ *hand-sketch*

- Compute the necessary values:
  - Period: $p = \dfrac{2\pi}{b}$
  - Phase shift: $\varphi = -\dfrac{c}{b}$
  - Fundamental interval: $[\varphi, \varphi + p)$

- Draw a number line with the values:



  (that way there is 1 period to the left and right of the "master tile")

- Draw the standard sine curve (should have 3 copies of the "master tile")

- Add the y-axis (possibly compute additional values such as $\frac{\varphi}{2}, \frac{\varphi}{4}$ for accurate placement).

- Add markings to y-axis for amplitude

Sinusoidal Graph ⇝ Equation

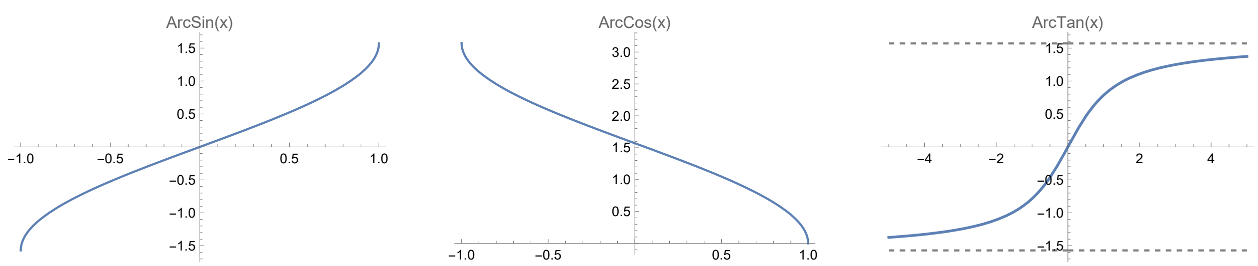
- Identify an individual wavelength, and read off its
  - Phase shift φ (= start)
  - Period p (= end − start)
  - Amplitude a

To read off, pick a window [start, end] of the graph that looks like a standard sine packet

- Compute the constants:
  - $b = \frac{2\pi}{p}$
  - $c = -\varphi \cdot b$

- Insert into the standard form of a sinusoidal function: $a \sin(bx + c)$

§2.6. *The Inverse Trigonometric Functions*



Domain and Range of inverse trig functions

$$\begin{aligned} \arcsin &: [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \arccos &: [-1, 1] \to [0, \pi] \\ \arctan &: (-\infty, \infty) \to (-\frac{\pi}{2}, \frac{\pi}{2}) \end{aligned}$$

Homework 6 # 4–6.
Slides 03/07

Probem-Type: Inverse trig of trig Example problems (from my recitation notes)

- $\arcsin\left(\sin \frac{11\pi}{4}\right) = \frac{\pi}{4}$
- $\arccos\left(\cos \frac{7\pi}{6}\right) = \frac{5\pi}{6}$
- $\arctan\left(\tan\left(\frac{5\pi}{6}\right)\right) = -\frac{\pi}{6}$

My approach to doing these problems is to know the unit circle well enough to do the inner trig evaluation, followed by knowing the inverse trig functions well enough to answer the resulting inverse-trig evaluation (the range for the trig functions may make the answer land in a different quadrant, but the reference angle should stay the same).

Nagy's approach is to use the below formulas to transform the angle so that the inversion identity applies.

Inversion identities:

$$\arcsin(\sin\theta) = \theta \qquad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$
$$\arccos(\cos\theta) = \theta \qquad \theta \in [0, \pi]$$
$$\arctan(\tan\theta) = \theta \qquad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

Negation in arctrig() identities:

$$\arcsin(-x) = -\arcsin x$$
$$\arccos(-x) = \pi - \arccos x$$
$$\arctan(-x) = -\arctan x$$

Negation in trig() identities:

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

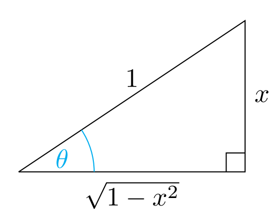
Invariance under full revolution of unit circle:

$$\text{trig}(\theta \pm 2\pi) = \text{trig}(\theta)$$

Trigonometric Functions of Inverses Use common sense to determine whether values are undefined (remember that the domain for both arcsin and arccos is $[-1, 1]$, and that division by zero is not allowed).

For $\theta = \arcsin x$:

Triangle corresponding to $\theta = \arcsin x$:

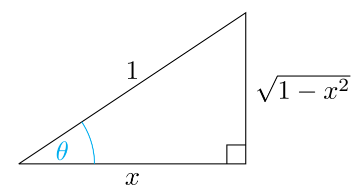


$$\sin(\arcsin x) = x \qquad \csc(\arcsin x) = \frac{1}{x}$$

$$\cos(\arcsin x) = \sqrt{1-x^2} \qquad \sec(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}} \qquad \cot(\arcsin x) = \frac{\sqrt{1-x^2}}{x}$$

Triangle corresponding to $\theta = \arccos x$:



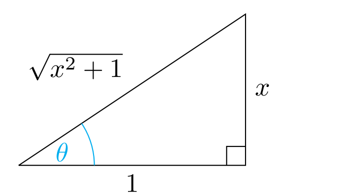
For $\theta = \arccos x$:

$$\sin(\arccos x) = \sqrt{1-x^2} \qquad \csc(\arccos x) = \frac{1}{\sqrt{1-x^2}}$$

$$\cos(\arccos x) = x \qquad \sec(\arccos x) = \frac{1}{x}$$

$$\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x} \qquad \cot(\arccos x) = \frac{x}{\sqrt{1-x^2}}$$

Triangle corresponding to $\theta = \arctan x$:



For $\theta = \arctan x$:

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}} \qquad \csc(\arctan x) = \frac{\sqrt{1+x^2}}{x}$$

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}} \qquad \sec(\arctan x) = \sqrt{1+x^2}$$

$$\tan(\arctan x) = x \qquad \cot(\arctan x) = \frac{1}{x}$$

§3.1 Applications to Vector Geometry

Length (aka magnitude) of a vector

If $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \end{bmatrix}$, then its length (aka magnitude) is given by the formula

$$\|\vec{\mathbf{v}}\| = \sqrt{x^2 + y^2}$$

Unit direction vector

The vector $\vec{\mathbf{u}}_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ is a unit vector in the direction given by θ. One can visualize the unit vectors as living on the unit circle (varying θ will trace out the unit circle).

Vector decomposition / reconstruction

A vector is comprised of two pieces of data: length and direction. As such, it can be decomposed into these two things. The length of a vector $\vec{\mathbf{v}}$ is extracted by the formula for $\|\vec{\mathbf{v}}\|$ written above. The vector's direction is extracted by normalizing the vector (rescaling the vector to make its length 1): $\vec{\mathbf{u}}_\theta = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$

The vector is reconstructed by taking a direction vector and scaling it to the appropriate length:

$$\vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|\vec{\mathbf{u}}_\theta$$

Dot and Skew product

For vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ we define the dot product and skew product:

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 := x_1 x_2 + y_1 y_2$$
$$\vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2 := x_1 y_2 - x_2 y_1$$

The skew product is like the cross product (but reduced down a dimension). It can be viewed as a determinant:

$$\vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2 = \det \begin{bmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \end{bmatrix} = \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

Properties of dot product and skew product

- Dot product is symmetric, Skew product is anti-symmetric [1]:

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_1$$
$$\vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2 = -\vec{\mathbf{v}}_2 \wedge \vec{\mathbf{v}}_1$$

- Both dot product and skew product satisfy linearity [2]:

$$\vec{\mathbf{v}}_1 \cdot (k\vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3) = k\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_3$$
$$\vec{\mathbf{v}}_1 \wedge (k\vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3) = k\vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_3$$

- Magnitude Identity: $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$
- $\vec{\mathbf{v}} \wedge \vec{\mathbf{v}} = 0$ (follows from anti-symmetry)

[1] this means that order will matter with skew product

[2] linearity combines both the distributivity and homogeneity conditions

Generalized Pythagoras Theorem

$$\|\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2\|^2 = \|\vec{\mathbf{v}}_1\|^2 + \|\vec{\mathbf{v}}_2\|^2 - 2(\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2)$$
$$\|\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2\|^2 = \|\vec{\mathbf{v}}_1\|^2 + \|\vec{\mathbf{v}}_2\|^2 + 2(\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2)$$

Relating dot and skew products with the angle between vectors

Nagy prefers thinking of angles as bidirectional (both clockwise and counterclockwise) and sets up the notions of

- turning angle τ of $\vec{\mathbf{v}}_1$ over $\vec{\mathbf{v}}_2$ [3]
- geometric angle γ between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$

[3] this is how much we need to rotate $\vec{\mathbf{v}}_1$ to lay it on top of $\vec{\mathbf{v}}_2$

The turning angle is defined to always be in the interval $(-\pi, \pi]$.

The geometric angle is always in the interval $[0, \pi]$.

An important property is that $\tau = \pm\gamma$ (positive when counterclockwise, negative when clockwise).

There are then two important sets of results:

Dot/Skew Product Theorem Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ be vectors. If either one of the vectors is zero, then

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2 = 0$$

Otherwise (both vectors are nonzero)

$$\begin{aligned} \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 &= \|\vec{\mathbf{v}}_1\| \cdot \|\vec{\mathbf{v}}_2\| \cdot \cos\tau \\ &= \|\vec{\mathbf{v}}_1\| \cdot \|\vec{\mathbf{v}}_2\| \cdot \cos\gamma \\ \vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2 &= \|\vec{\mathbf{v}}_1\| \cdot \|\vec{\mathbf{v}}_2\| \cdot \sin\tau \end{aligned}$$

Note that the reason for the additional rule for dot product is because $\cos(-x) = \cos(x)$.

Vector Angle Formulas Assume $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are nonzero vectors. The turning angle τ of $\vec{\mathbf{v}}_1$ over $\vec{\mathbf{v}}_2$ is the unique angle in the interval $(-\pi, \pi]$ that satisfies the equalities

$$\begin{cases} \cos\tau = \dfrac{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2}{\|\vec{\mathbf{v}}_1\| \cdot \|\vec{\mathbf{v}}_2\|} \\ \sin\tau = \dfrac{\vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2}{\|\vec{\mathbf{v}}_1\| \cdot \|\vec{\mathbf{v}}_2\|} \end{cases}$$

The geometric angle γ between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ is given by the formula:

$$\gamma = \arccos\left(\frac{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2}{\|\vec{\mathbf{v}}_1\| \cdot \|\vec{\mathbf{v}}_2\|}\right)$$

The turning angle τ can be obtained from the geometric angle γ using:

$$\tau = \operatorname{sign}(\vec{\mathbf{v}}_1 \wedge \vec{\mathbf{v}}_2)\gamma$$

(By convention, sign(0) = +. This is to handle the situation when $v_2 = -v_1$.)

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Post Midterm 3 material

Orthogonality

Two vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are *perpendicular* if and only if their dot product is 0.

Two vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are *parallel* if and only if their skew product is 0.

Projection

The projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{w}}$ is

$$\operatorname{proj}_{\vec{\mathbf{w}}}(\vec{\mathbf{v}}) := \left(\frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\vec{\mathbf{w}} \cdot \vec{\mathbf{w}}}\right)\vec{\mathbf{w}}$$

(3.1.22) in Nagy's textbook. He also gives two other expressions for it, but this form best showcases how to compute it.

Area of parallelogram and triangle

Given two vectors $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, the area of the parallelogram $\mathscr{P}$ formed by them is given by

$$\text{Area}(\mathscr{P}) = |\vec{\mathbf{v}} \wedge \vec{\mathbf{w}}|$$

The area of a triangle $\mathscr{T}$ formed by them will be half that of the parallelogram:

To see this visually: Cut the parallelogram in half by drawing a diagonal

$$\text{Area}(\mathscr{T}) = \frac{1}{2}|\vec{\mathbf{v}} \wedge \vec{\mathbf{w}}|$$

§3.2. Applications to Triangle Geometry

Law of Cosines

Let $\triangle ABC$ be a triangle with the standard labeling for angles (angle $\widehat{A}$ is opposite side a, etc.)

The law of cosines says:

$$c^2 = a^2 + b^2 - 2ab\cos\widehat{C}$$
$$b^2 = a^2 + c^2 - 2ac\cos\widehat{B}$$
$$a^2 = b^2 + c^2 - 2bc\cos\widehat{A}$$

These equations are the same, up to permuting letters.

Law of Sines

The law of sines says:

$$\frac{\sin\widehat{A}}{a} = \frac{\sin\widehat{B}}{b} = \frac{\sin\widehat{C}}{c}$$

Problem-type: Triangle solving

A triangle has six pieces of data (3 sides, 3 angles). In a problem of this type, 3 pieces of data will be given, and finding the other 3 pieces is the objective.

Only five cases are possible:

1. SSS. Use the rearranged version of Law of Cosines to find one of the missing angles. Then use the Law of Sines to find the other two angles. (Warning: This case can have no solution. This manifests when the fraction expression inside the arccosine is outside the interval $[-1, 1]$. Plugging such an expression into a calculator should give an error, or a complex number.)

Rearranged Law of Cosines:

$$\widehat{C} = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$
$$\widehat{B} = \arccos\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$
$$\widehat{A} = \arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

2. SAS. Insert values into law of cosines to get the length of the missing side. Then use law of sines to get the other two angles.

3. ASA. Calculate the missing angle by subtracting from 180°. Then use law of sines to find the other two sides.
4. AAS. Calculate the missing angle by subtracting from 180°. Then use law of sines to find the other two sides.
5. SSA. This is the hard case. There can be 2, 1, or 0 solutions.
   I. Compare the given sides and figure out the order of the angles facing them.
   II. Set of the Law of Sines, and find the value of the sine of the missing angle that faces one of the given sides. If value > 1, stop. The problem has no solutions.
   III. Using the value found in Step II, find the missing angle, which can be either of:

   $$\begin{aligned} \text{angle}_1 &= \arcsin(value) \\ \text{angle}_2 &= 180^\circ - \arcsin(value) \end{aligned}$$

   Using the information from Step I, decide whether one, or both angles are acceptable.
   IV. Each of the angles found in Step III gives an AAS Problem. Solve each of these as outlined earlier:
      i. Find the third angle by subtracting both known angles from 180°.
      ii. Use Law of Sines to find the missing side.

Area Formulas

If the data given for the triangle doesn't match SAS or Heron, you'll need to partially solve the triangle

The SAS Area Formula

$$\text{Area}(\triangle ABC) = \frac{ab\sin\widehat{C}}{2} = \frac{ac\sin\widehat{B}}{2} = \frac{bc\sin\widehat{A}}{2}.$$

Heron's Formula For a triangle $\mathcal{T}$ with sides a, b, c,

$$\text{Area}(\mathcal{T}) = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter of $\mathcal{T}$, that is,

$$s = \frac{a+b+c}{2}.$$

§4.1. Sums and Differences of Angles

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \end{aligned}$$

Sum of sine and cosine with same frequency

The expression

$$E(x) = a\cos kx + b\sin kx$$

can be rewritten as

$$\sqrt{a^2+b^2}\cos(kx-\tau)$$

where $\tau = (\text{sign of } b)\arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$

§4.2. Multiples of Angles

Double Angle Formulas

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha \\ &= \frac{2\tan\alpha}{1+\tan^2\alpha} \\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha \\ &= \frac{1-\tan^2\alpha}{1+\tan^2\alpha} \\ \tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}\end{aligned}$$

Triple Angle Formulas

$$\begin{aligned}\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha \\ \cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha \\ \tan 3\alpha &= \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}\end{aligned}$$

Half Angle Formulas Warning! In the below formulas, the '$\pm$' is an ambiguity that you need to resolve by using the quadrant the angle $\frac{\theta}{2}$ is in, and whether the trig function is + or $-$ in that quadrant.

These can be derived from the double angle formulas

$$\begin{aligned}\sin(\theta/2) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\ \cos(\theta/2) &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\ \tan(\theta/2) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\ &= \frac{\sin\theta}{1+\cos\theta} \\ &= \frac{1-\cos\theta}{\sin\theta}\end{aligned}$$

§4.3. *From Products to Sums and Back*

Product-to-Sum Identities

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

Sum-to-Product Identities

$$\cos u+\cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\cos u-\cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$
$$\sin u+\sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\sin u-\sin v = 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)$$

Complex Numbers

There are two ways to represent a complex number:

- Cartesian: $z = a + bi$
- Polar: $z = r(\cos\theta + i\sin\theta)$ which is abbreviated as $r\,\mathrm{cis}\,\theta$

Polar also has an exponential format: $z = re^{i\theta}$

Converting Cartesian ⇜⇝ Polar

To convert from Polar to Cartesian, just evaluate the trig and distribute the r.

To convert from Cartesian (x, y) to Polar (r, θ), use the following:

$$r = \sqrt{x^2+y^2}$$
$$\tau = (\text{sign of } y)\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

You may have seen / used $\theta = \arctan\left(\frac{y}{x}\right)$. This formula is technically better.

The full set of representations will have $\theta = \tau + 2\pi k$, where $k \in \mathbb{Z}$. The ***principal representation*** is the one with $\theta \in (-\pi, \pi]$. (Just let $k = 0$)

De Moivre's formula

This is an easy way to raise a complex number to an integer power.

If $z = r\,\mathrm{cis}\,\theta$, then

$$z^n = r^n\,\mathrm{cis}(n\theta).$$

Roots of a complex number

If $w = r \operatorname{cis} \theta$, then the n-th roots of w are

$$z_k = \sqrt[n]{r} \operatorname{cis} \left(\frac{\theta + 2k\pi}{n} \right) \qquad (k = 0, 1, \ldots, n-1)$$