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Sur les A_∞-catégories (translated)

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I would like this line to be read distinctly, repeated with conviction, in a breath of gratitude. I want it to be imprinted, a reminder of the measured distance that underlies the respect we have for each other. Hallelujah! The chance that brought us together was a beautiful journey.

Well, if these acknowledgments don't fit into one line, it's because each, and I emphasize this fact, deserves the line that is theirs.

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And then, a conclusion to these lines, an impossible end: there is a specific point in my heart that is occupied by those I love and those who love me (sometimes disjointed individuals). This place cannot be described here... Three ellipsis points, somersaults, peanuts, and then they're gone!

Abstract/Résumé

Abstract

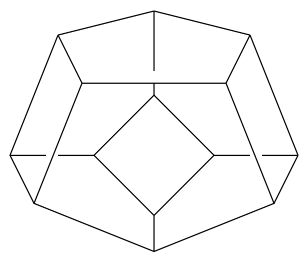
We study (not necessarily connected) $\mathbf{Z}$-graded A_∞-algebras and their A_∞-modules. Using the cobar and the bar construction and Quillen's homotopical algebra, we describe the localisation of the category of A_∞-algebras with respect to A_∞-quasi-isomorphisms. We then adapt these methods to describe the derived category $\mathcal{D}_\infty A$ of an augmented A_∞-algebra A. The case where A is not endowed with an augmentation is treated differently. Nevertheless, when A is strictly unital, its derived category can be described in the same way as in the augmented case. Next, we compare two different notions of A_∞-unitality : strict unitality and homological unitality. We show that, up to homotopy, there is no difference between these two notions. We then establish a formalism which allows us to view A_∞-categories as A_∞-algebras in suitable monoidal categories. We generalize the fundamental constructions of category theory to this setting : Yoneda embeddings, categories of functors, equivalences of categories... We show that any algebraic triangulated category $\mathcal{T}$ which admits a set of generators is A_∞-pretriangulated, that is to say, $\mathcal{T}$ is equivalent to $H^0\mathsf{tw}\mathcal{A}$, where $\mathsf{tw}\mathcal{A}$ is the A_∞-category of twisted objects of a certain A_∞-category $\mathcal{A}$.

Thus we give detailed proofs of a part of the results on homological algebra which M. Kontsevich stated in his course "Triangulated categories and geometry" [Kon98].

Résumé

Nous étudions les A_∞-algèbres $\mathbf{Z}$-graduées (non nécessairement connexes) et leurs A_∞-modules. En utilisant les constructions bar et cobar ainsi que les outils de l'algèbre homotopique de Quillen, nous décrivons la localisation de la catégorie des A_∞-algèbres par rapport aux A_∞-quasi-isomorphismes. Nous adaptons ensuite ces méthodes pour décrire la catégorie dérivée $\mathcal{D}_\infty A$ d'une A_∞-algèbre augmentée A. Le cas où A n'est pas muni d'une augmentation est traité différemment. Néanmoins, lorsque A est strictement unitaire, sa catégorie dérivée peut être décrite de la même manière que dans le cas augmenté. Nous étudions ensuite deux variantes de la notion d'unitarité pour les A_∞-algèbres : l'unitarité stricte et l'unitarité homologique. Nous montrons que d'un point de vue homotopique, il n'y a pas de différence entre ces deux notions. Nous donnons ensuite un formalisme qui permet de définir les A_∞-catégories comme des A_∞-algèbres dans certaines catégories monoïdales. Nous généralisons à ce cadre les constructions fondamentales de la théorie des catégories : le foncteur de Yoneda, les catégories de foncteurs, les équivalences de catégories... Nous montrons que toute catégorie triangulée algébrique engendrée par un ensemble d'objets est A_∞-prétriangulée, c'est-à-dire qu'elle est équivalente à $H^0\mathsf{tw}\mathcal{A}$, où $\mathsf{tw}\mathcal{A}$ est l'A_∞-catégorie des objets tordus d'une certaine A_∞-catégorie $\mathcal{A}$.

Nous démontrons ainsi une partie des énoncés d'algèbre homologique presentés par M. Kontsevich pendant son cours "Catégories triangulées et géométrie" [Kon98].



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Introduction

We refer to [Kel01a] and [Kel01b] for an introduction to A_∞-algebras and their modules. This thesis contains, among other things, the detailed proofs of the statements of [Kel01a]. Apart from [Kon98] and [Kel01a], we relied mainly on the article by V. Hinich [Hin01] and on the following works: [Sta63a], [Pro85], [GJ90], [HK91], [GLS91], [Mar96], [Hin97]. Some of the results of this thesis have been obtained recently and independently by K. Fukaya [Fuk01a], P. Seidel [Sei], A. Lazarev [Laz02], V. Lyubashenko [Lyu02] and M. Kontsevich and Y. Soibelmann [KS02b], [KS02a].

Strict structures and structures up to homotopy

The so-called strict structures of classical algebra, for example associative, commutative or Lie algebras, have proved to be insufficient in topology because they are not compatible with homotopy. Thus, if X is a loop space and Y is a topological space homotopic to X, it is not always possible to transfer the H-space structure (which is strict) from X to Y. It is to overcome this defect that J. Stasheff [Sta63a] introduced the notion of A_∞ structure, which is a relaxation of that of topological semigroup. The A_∞ structures are part of the *structures up to homotopy*, that is to say structures whose "lack of strictness" is controlled in a coherent way by *homotopies of higher order*. For some structures up to homotopy, higher order homotopies have been known for a long time like the Steenrod operations [Ste47], [Ste52] or the Massey products. Structures up to homotopy behave well with respect to homotopy equivalences: if an object (topological or differential graded) is provided with a structure up to homotopy, one can under certain conditions translate it onto another object when the latter is homotopic to the starting object. The first part of this thesis will deal with algebraic A_∞-structures, that is to say A_∞-structures in the framework of differential graded algebra. In the second part we will study their generalizations to the categorical framework.

Algebraic A_∞-structures

Let $\mathbb{K}$ be a field. An A_∞-*algebra* [Sta63b] is a $\mathbf{Z}$-graded $\mathbb{K}$-vector space A endowed with graded morphisms

$$m_i : A^{\otimes i} \to A, \quad i \geq 1,$$

of degree $2-i$, satisfying equations of which the first says that m_1 is a differential, the second that m_1 is a derivation for the *multiplication* m_2 and the third

$$m_2(m_2 \otimes \mathbf{1}) - m_2(\mathbf{1} \otimes m_2) = \delta(m_3)$$

that the lack of associativity of m_2 is measured by the boundary of m_3 in the differential graded space $\mathsf{Hom}(A^{\otimes 3}, A)$. Intuitively, an A_∞-algebra is therefore a "differential graded algebra whose lack of associativity is controlled (in the strong sense) by homotopies of higher order". If A and A' are two A_∞-algebras, an A_∞*-morphism* $f : A \to A'$ is a sequence of graded morphisms

$$f_i : A^{\otimes i} \to A', \quad i \geq 1,$$

of degree $1-i$, satisfying equations the first of which assert that f_1 is a morphism of complexes which is compatible with the multiplications m_2 and m_2' up to a homotopy f_2. In the same way, if f and g are A_∞-morphisms $A \to A'$, a *homotopy* h between f and g is a sequence of morphisms

$$h_i : A^{\otimes i} \to A', \quad i \geq 1,$$

of degree $-i$, which satisfy equations of which the first two assert that h_1 is a homotopy between the "morphisms of differential graded algebras"

$$f_1 \quad \text{and} \quad g_1 : (A, m_1, m_2) \to (A', m_1', m_2').$$

Let A be an A_∞-algebra. An A*-polydule* (called A_∞*-module over* A in the literature) is a $\mathbf{Z}$-graded $\mathbb{K}$-vector space M endowed with graded morphisms

$$m_i^M : M \otimes A^{\otimes i-1} \to M, \quad i \geq 1,$$

of degree $2-i$, satisfying equations whose first affirm that m_1^M is a differential and that m_2^M defines an action of the (strongly homotopically) associative algebra A whose compatibility with the multiplication of A is controlled by the higher order homotopy m_3^M. As for A_∞-algebras, we have A_∞-morphisms between A-polydules and homotopies between A_∞-morphisms.

Link to operad theory

Some arguments of the thesis are related to the theory of operads (for example, the obstruction theory of A_∞-algebras (B.1)). We will not explicitly use the operad formalism in our statements (and their proofs), preferring a naive approach. Let us nevertheless recall some facts and references on this subject.

The Stasheff cell complexes $\{K_i \times \Sigma_i\}_{i\geq 2}$ (see [Sta63a]) form a topological operad [May72]. The chain complexes associated with them therefore form a differential graded operad. This is an operad of A_∞-algebras. Differential graded operads were extensively studied in the early 90s [HS93], [GJ94], [GK94] to clarify the link between strict structures and structures up to homotopy [GK94], [Mar96], [Mar99], [Mar00]. With regard to the A_∞ structures, we will retain two results from the operads: *the operad of* A_∞*-algebras is the minimal cofibrant model in the sense of M. Markl [Mar96] of the operad of associative algebras* Ass; *the Koszul dual* $\mathsf{Ass}^!$ *of* Ass *is the co-operad of co-associative coalgebras.*

Chapter 1 : a homotopy theory of A_∞-algebras

First recall a result from H. J. Munkholm. Let DA be the category of differential graded algebras (satisfying certain conditions on the grading and on the connectedness) and $\mathsf{Ho}\,\mathsf{DA}$ the localization of DA with respect to quasi-isomorphisms. Let DASH be the category of differential graded algebras whose morphisms are the A_∞-morphisms. Using the ideas of J. Stasheff and S. Halperin [SH70], H. J. Munkholm [Mun78] (see also [Mun76]) showed, first, that *the homotopy relation on* $\mathsf{Hom}_{\mathsf{DA}}(A, A')$, $A, A' \in DA$, *(which is not an equivalence relation in general) extends to a relation* *over the morphism spaces* $\mathsf{Hom}_{\mathsf{DASH}}(A, A')$ *which is an equivalence relation for any* A *and* A'[1], and secondly, that *the category* $\mathsf{Ho}\,\mathsf{DA}$ *is equivalent to the quotient of* DASH *by this equivalence relation.* In other words, even if it means increasing the number of morphisms between differential graded algebras, localization with respect to quasi-isomorphisms is equivalent to passing to the quotient with respect to homotopy. In the first part of this chapter, we will generalize the results of [Mun78] to A_∞-algebras. An A_∞*-quasi-isomorphism* f is an A_∞-morphism such that f_1 is a quasi-isomorphism. We show the following results:

(Homotopy theorem) *The homotopy relation on* A_∞*-morphisms is an equivalence relation (1.3.1.3 a).*

(Theorem of A_∞-quasi-isomorphisms) *Every* A_∞*-quasi-isomorphism of* A_∞*-algebras is invertible up to homotopy (1.3.1.3 b).*

The topological analogue of the A_∞-quasi-isomorphism theorem is due to M. Fuchs [Fuc76] (see also [Fuc65]). In his thesis [Pro85], A. Prouté proved the two theorems under conditions of scaling or connectedness (see also [Kad87]). The need to generalize these results is due to the fact that in the constructions of K. Fukaya et al. of A_∞-algebras (A_∞-categories), nonzero components can appear in any integer degree. In the general case, we will deduce the above theorems from the following results: *the bar construction* B *is an equivalence of categories between* Alg_∞*, the category of* A_∞*-algebras, and the subcategory of cofibrant and fibrant objects of a model category* Cogc *of coalgebras (1.3.1.2). The bar construction matches the homotopy of* A_∞*-morphisms to the left homotopy of* Cogc *between morphisms between cofibrant and fibrant objects (1.3.4.1) and the* A_∞*-quasi-isomorphisms to weak equivalences (1.3.3.5).* [2]

The category Cogc in question is the category of cocomplete differential graded coalgebras. Let Alg be the category of differential graded algebras and $\Omega : \mathsf{Cogc} \to \mathsf{Alg}$ the cobar construction. The model category structure of Cogc (1.3.1.2. a) is such that the pair of adjoint functors

$$(\Omega, B) : \mathsf{Cogc} \to \mathsf{Alg},$$

is a Quillen equivalence (1.3.1.2. b). The use of this pair of adjoint functors to study the category $\mathsf{Alg}[Qis^{-1}]$ dates back to the 1970s with the work of D. Husemoller, J. C. Moore and J. Stasheff [HMS74] (see also [EM66]). They consider augmented positively graded differential algebras, on the one hand and co-augmented positively graded and connected differential coalgebras, on the other hand, and show that the localization of the category of algebras with respect to quasi-isomorphisms is equivalent to the location of the category of coalgebras with respect to quasi-isomorphisms. Without the assumptions about graduation or connectedness, their statement is no longer true. In the general case (1.3.1.2), we must replace the class of quasi-isomorphisms of Cogc by a class of morphisms (called weak equivalences) which is strictly contained in that of quasi-isomorphisms (1.3.5.1. c). We show that between two positively graded coalgebras, the weak equivalences are exactly the quasi-isomorphisms (see 1.3.5.1. e). Our results thus generalize [HMS74, Chap. II, Thm. 4.4 and Thm. 4.5].

Our proof that Cogc admits a model category structure (1.3.1.2) follows the ideas of V. Hinich [Hin01] inspired by those of Quillen [Qui67], [Qui69]. We lift the model category structure of Alg

[1] this might be mistranslated. "la relation d'homotopie sur $\mathsf{Hom}_{\mathsf{DA}}(A, A')$, $A, A' \in DA$, (qui n'est pas une relation d'équivalence en général) s'étend en une relation sur les espaces de morphismes $\mathsf{Hom}_{\mathsf{DASH}}(A, A')$ qui est une relation d'équivalence quelles que soient A et A'"

[2] Not sure how to translate this.. "La construction bar fait correspondre l'homotopie des A_∞-morphismes à l'homotopie à gauche de Cogc entre morphismes entre objets cofibrants et fibrants (1.3.4.1) et les A_∞-quasi-isomorphismes aux équivalences faibles (1.3.3.5)."

along the adjunction (Ω, B). This adjunction is of the same type as the adjunction between the category of differential graded Lie algebras and the category of cocommutative differential graded coalgebras in rational homotopy. It comes from the Koszul duality between the operad Ass and the co-operad of co-associative coalgebras.

The characterization of fibrant objects of Cogc can be interpreted as a consequence of the fact that the operad of A_∞-algebras is the minimal cofibrant model in the sense of M. Markl [Mar96] of the operad of associative algebras. This fact implies that the obstruction to the construction by induction of the graded morphisms m_i, $i \geq 1$, defining a A_∞-structure on a graded object A is of the form "m_{n+1} must kill a certain cocycle (built from m_i, $1 \leq i \leq n$)" (see B.1.2). The condition which measures the obstruction to the construction by recurrence of A_∞-morphisms is of the same type (B.1.5). We call the study of these obstructions the *obstruction theory* of A_∞-algebras. This theory is the subject of the appendix (B.1).

At the end of chapter 1, we will re-prove (1.4.1.1) the "compatibility of homotopic A_∞ structures": *let A be an A_∞-algebra and*

$$g : (V, d) \to (A, m_1^A)$$

a homotopy equivalence of complexes. There exists an A_∞-algebra structure on V such that m_1^V is equal to d and such that V and A are homotopic as A_∞-algebras. This result is well known. T. Kadeishvili [Kad80] and A. Prouté [Pro85] showed it in the case where $d = 0$ and under assumptions on scaling and connectedness using obstruction methods. The general case is due to V. K. A. M. Gugenheim, L. A. Lambe and J. Stasheff [GLS91] who use the "tensor trick" invented by J. Huebschmann [Hue86]. The essential point of their proof is that the perturbation lemma [Gug72] is compatible with an additional structure (coalgebra in our case). On this subject, see also [HK91], [GL89] and the historical reminders of the section 1.4. Our proof of "homotopic compatibility" (section 1.4.1.1) will be based on obstruction theory (B.1). "Homotopy compatibility" implies that every A_∞-algebra A admits a *minimal model*, i.e., an A_∞ structure on the homology H^*A such that H^*A and A are homotopic as A_∞-algebras (1.4.1.4). The link between a certain minimal model obtained by our method and that obtained by the perturbation lemma [GLS91] is described in (1.4.2.1).

The "minimality" of the model H^*A above refers to the fact that the tensor coalgebra $B(H^*A)$ is a minimal model (in the sense of H. J. Baues and J.-M. Lemaire [BL77]) of the coalgebra BA.

Chapter 2 : a homotopy theory of polydules

Let A be an *augmented* A_∞-algebra. Recall that in this thesis the structures commonly called A_∞-modules over A are called *A-polydules* ("poly" because the structure is equipped with several multiplications).

The purpose of this chapter is to describe the derived category $\mathcal{D}_\infty A$ whose objects are the strictly unital A-polydules. We adapt for this the methods of homotopic algebra in chapter 1 to A-polydules. The derived category from an A_∞-algebra which is not endowed with an augmentation will be studied in chapter 4.

Let C be a cocomplete co-augmented differential graded coalgebra and $\mathsf{Comc}\, C$ the category of cocomplete counital differential graded C-comodules. We construct (2.2.2.2) a model category structure on $\mathsf{Comc}\, C$ which is such that, if A is an augmented differential graded algebra and $\tau : C \to A$ an admissible acyclic twisting cochain, the couple of adjoint functors "twisted tensor products" (2.2.1)

$$(- \otimes_\tau A, ? \otimes_\tau C) : \mathsf{Comc}\, C \to \mathsf{Mod}\, A$$

is a Quillen equivalence. The "twisted tensor product" functors here replace the bar and cobar constructions of the previous chapter. The homotopy category $\mathsf{Ho}\,\mathsf{Comc}\,C$ (see appendix A) is therefore equivalent to the derived category

$$\mathcal{D}A = \mathsf{Ho}\,\mathsf{Mod}\,A.$$

In [HMS74], D. Husemoller, J. C. Moore and J. Stasheff proved a slightly more general result (Theorem 5.15) but under assumptions on scaling and connectedness. We will not consider extended algebras and coalgebras here (see [HMS74]), restricting ourselves to the separate study of (co)algebras and their (co)modules. Let us just notice that our result (2.2.2.2) generalizes the specialization of theorem 5.15 of [HMS74] to the subcategory formed by the extended algebras $(M, A, 0)$, where A is a fixed algebra and M an A-module, and in its image in the category of extended coalgebras.

We then study the fibrant objects of $\mathsf{Comc}\,C$ for a certain class of coalgebras C. Let A be an augmented A_∞-algebra and $\mathsf{Mod}_\infty\,A$ the category of strictly unital A-polydules whose morphisms are the strictly unital A_∞-morphisms. Let B^+A be the bar construction co-augmented by the reduction $\overline{A}$ of A. When C is a coalgebra isomorphic to B^+A, we show (2.4.1.3) using obstruction theory (B.2) that *an object of* $\mathsf{Comc}\,C$ *is fibrant if and only if it is a direct factor of an almost cofree object*. As all the objects of $\mathsf{Comc}\,C$ are cofibrant, the subcategory of cofibrant and fibrant objects is the essential image of the bar construction of strictly unital A-polydules. We deduce (2.4.2.2) that the derived category

$$\mathcal{D}_\infty A = \mathsf{Mod}_\infty\,A[\{Qis\}^{-1}]$$

is equivalent to the quotient of the category $\mathsf{Mod}_\infty\,A$ by the homotopy relation (this proves the theorem of A_∞-quasi-isomorphisms for A-polydules). The triangulated structure of $\mathcal{D}_\infty A$ will be studied in section (2.4.3).

In the section 2.5, we study, by the same methods, the derived category of strictly unital bipolydules (called A_∞*-bimodules* in the literature) over two augmented A_∞-algebras. We will use the results of this section in the second part of the thesis which concerns A_∞-categories.

Chapter 3 : Units

An associative $\mathbb{K}$-algebra (A, μ) is *unital* if it is equipped with a morphism $\eta : \mathbb{K} \to A$ satisfying the relations

$$\mu(\eta \otimes \mathbf{1}) = \mathbf{1} \quad \text{and} \quad \mu(\mathbf{1} \otimes \eta) = \mathbf{1}.$$

There are several generalizations of the notion of unitality to A_∞-algebras. We study two of them: *strict unitality* (already present in the topological version of J. Stasheff [Sta63a]) and *homological unitality*. Strict unitality is the notion that will allow us to generalize certain classical properties of unital algebras to A_∞-algebras. The more general homological unitality appears in the geometric examples [Fuk93]. We show that from a homotopy point of view there is no difference between these two possible generalizations of the notion of unitality. More precisely, we will show the following result: let $\left(\mathsf{Alg}_\infty\right)_{hu}$ be the category of homologically unital A_∞-algebras whose morphisms are the homologically unital A_∞-morphisms and $\left(\mathsf{Alg}_\infty\right)_{su}$ the category of strictly unital A_∞-algebras whose morphisms are the strictly unital A_∞-morphisms. *The categories* $\left(\mathsf{Alg}_\infty\right)_{hu}$ *and* $\left(\mathsf{Alg}_\infty\right)_{su}$ *become equivalent after passing to homotopy* (3.2.4.4). The proof of this result will be based on an obstruction theory of minimal A_∞-structures (B.4) and on the existence of a strictly unital minimal model for strictly unital A_∞-algebras (3.2.4.1).

Recently, K. Fukaya [FOOO01], [Fuk01b], P. Seidel [Sei], A. Lazarev [Laz02] and V. Lyubashenko [Lyu02] have studied the problem of units independently. V. Lyubashenko's generalization of the notion of unitality specializes to our notion of homological unitality if we work over a field (V. Lyubashenko works over any commutative ring).

Chapter 4 : the derived category

Here, we define the derived category from an arbitrary A_∞-algebra A (not necessarily strictly unital). We will show that, when A is strictly unital, its derived category admits the following four descriptions (4.1.3.1):

D1. the triangulated subcategory $\mathsf{Tria}\, A$ of the derived category $\mathcal{D}_\infty(A^+)$ (where A^+ is the augmentation of A and $\mathcal{D}_\infty(A^+)$ is defined in chapter 2),

D2. the category
$$\mathcal{H}_\infty A = \mathsf{Mod}_\infty A/\sim$$
where $\mathsf{Mod}_\infty A$ is the category of strictly unital A-polydules and $\sim$ is the homotopy relation,

D3. the localized category
$$\big(\mathsf{Mod}_\infty A\big)[Qis^{-1}]$$
where Qis is the class of A_∞-quasi-isomorphisms of $\mathsf{Mod}_\infty A$,

D4. the localized category
$$\big(\mathsf{Mod}_\infty^{\mathsf{strict}} A\big)[Qis^{-1}]$$
where $\mathsf{Mod}_\infty^{\mathsf{strict}} A$ is the non-full subcategory of $\mathsf{Mod}_\infty A$ whose morphisms are the strict A_∞-morphisms.

We will show (4.1.3.8) that if A is a unital differential graded algebra, the derived category $\mathcal{D}A$ (see for example [Kel94a]) is equivalent to the categories defined above.

Chapter 5 : preliminaries on A_∞-categories.

The notion of A_∞-category is a natural generalization of that of A_∞-algebra. At the beginning of the 1990s, the work of K. Fukaya [Fuk93] (see also [Fuk01b]) showed that it appears naturally in the study of Floer homology. Inspired by these works, M. Kontsevich, in his talk [Kon95] at the international congress, gave a conjectural interpretation of mirror symmetry as the "shadow" of an equivalence between the derived categories of two A_∞-categories of geometric origin (see also [PZ98] where this conjecture was demonstrated for elliptic curves). In the rest of this thesis, we generalize to the A_∞-categorical framework the fundamental constructions of category theory: the Yoneda functor, the categories of functors, the equivalences of categories, etc., and prove some of the results stated or implied in [Kon98]. For this, we will use or adapt certain methods from the first part of the thesis.

An A_∞-category is an A_∞-algebra with several objects, and conversely, an A_∞-algebra is a A_∞-category with one object. The problems raised by the increase in the number of objects are numerous and the generalization of the results of the previous chapters is sometimes very technical (for example for the homotopy between A_∞-morphisms). We introduce a bicategory C whose objects are sets. As C is a bicategory, for any set $\mathbb{O}$, the category of morphisms $\mathsf{C}(\mathbb{O},\mathbb{O})$ is a monoidal category (see [ML98, Chap. XII, §6]). We define (5.1.2.1) a small A_∞-category whose set of objects is in bijection with a set $\mathbb{O}$ as an A_∞-algebra in $\mathsf{C}(\mathbb{O},\mathbb{O})$. We then define the A_∞-functors and the differential graded categories $\mathcal{C}_\infty\mathcal{A}$ and $\mathcal{C}_\infty(\mathcal{A},\mathcal{B})$ of $\mathcal{A}$-polydules and $\mathcal{A}$-$\mathcal{B}$-strictly unital bipolydules ($\mathcal{A}$ and $\mathcal{B}$ are strictly unital A_∞-categories). A key lemma which will be useful for the construction of the Yoneda A_∞-functor (chapter 7) is demonstrated in (5.3.0.1).

Chapter 6 : the torsion of A_∞-categories.

In this chapter, we generalize to A_∞-algebras a torsion technique well known in deformation theory (for an overview, see for example [Hue99]). Let $\mathcal{A}$ be an A_∞-category. Consider the generalized Maurer-Cartan equation

$$\sum_{i=1}^{\infty} m_i\big(x\otimes\ldots\otimes x\big)=0.$$

We show (6.1.2 and 6.2.4) that a solution x of this equation (when it makes sense) gives a new A_∞-category $\mathcal{A}_x$ called the twist of $\mathcal{A}$ by x. The twist of A_∞-algebras is due to K. Fukaya who introduced it (as well as that of L_∞-algebras) in [Fuk01b] and [Fuk01a] for the study of A_∞-deformations. Our formulas for the twisted compositions m_i^x, $i\geq 1$, of $\mathcal{A}_x$ are the same (except for equivalent signs) as in [Fuk01b] but the proof that they well-define an A_∞-category structure is different. We then describe the torsion of A_∞-functors (6.1.3 and 6.2.5) and of (bi)polydules (6.1.4 and 6.2.6) by solutions of the Maurer-Cartan equation. We also show that if an A_∞-functor f induces a quasi-isomorphism on the morphism spaces, its torsion f_x also induces a quasi-isomorphism on the morphism spaces (6.1.3.4).

The twist will be useful in chapters 7 and 8.

Chapter 7 : the A_∞-Yoneda functor and twisted objects.

Let $\mathcal{A}$ be a category. Recall that the Yoneda functor is the functor

$$\mathcal{A}\to\mathsf{Mod}\,\mathcal{A},\quad A\mapsto\mathsf{Hom}_{\mathcal{A}}(-,A).$$

In this chapter, we raise this functor into a A_∞-functor (7.1.0.1)

$$y:\mathcal{A}\to\mathcal{C}_\infty\mathcal{A},\quad A\mapsto\mathsf{Hom}_{\mathcal{A}}(-,A),$$

where $\mathcal{A}$ is an A_∞-category. If $\mathcal{A}$ is strictly unital, we show that the A_∞-functor y is strictly unital and that it is factorized by the A_∞-category of twisted objects $\mathsf{tw}\mathcal{A}$ (7.1.0.4). The compositions of the A_∞-category $\mathsf{tw}\mathcal{A}$ are obtained by torsion (chapter 6). If $\mathcal{G}$ is a unital differential graded category, the (differential graded) category of twisted objects is due to A. I. Bondal and M. M. Kapranov [BK91] (they notate it $\mathrm{Pr}\text{-}\mathrm{Tr}^+\mathcal{G}$). The purpose of [BK91] is to overcome a deficiency of the axioms of triangulated categories to describe derived categories [Ver77]: the cone is not functorial. Rather than triangulated categories, they consider pre-triangulated categories described using the category of twisted objects and show the following equivalence of categories: let $\mathcal{E}$ be a pre-triangulated category ($H^0\mathcal{E}$ is then triangulated). Let $\mathcal{G}$ be a full subcategory of $\mathcal{E}$. The triangulated subcategory $\mathsf{tria}\,\mathcal{G}\subset H^0\mathcal{E}$ generated by $\mathcal{G}$ is equivalent to the triangulated category $H^0(\mathrm{Pr}\text{-}\mathrm{Tr}^+\mathcal{G})$. In the A_∞ case, we have the same results: we show (7.4) that if $\mathcal{A}$ is a strictly unital A_∞-category, the categories

$$H^0\mathsf{tw}\mathcal{A}\quad\text{and}\quad\mathsf{tria}\,\mathcal{A}\subset\mathcal{D}_\infty\mathcal{A}$$

are equivalent (as stated in [Kon95]). Moreover, we show (section 7.6) that any algebraic triangulated category which is generated by a set of objects is A_∞-pre-triangulated, i.e., it is equivalent to $H^0\mathsf{tw}\mathcal{A}$, for some A_∞-category $\mathcal{A}$.

Let $\mathcal{A}$ be a strictly unital A_∞-category. The category $\mathcal{C}_\infty\mathcal{A}$ is differential graded and the A_∞-Yoneda functor $y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$ induces (7.4.0.1) a quasi-isomorphism on morphism spaces. We deduce that the image $y(\mathcal{A}) \subset \mathcal{C}_\infty\mathcal{A}$ is a unital differential graded category which is quasi-isomorphic to $\mathcal{A}$. This shows that from a homological point of view, the study of strictly unital A_∞-categories (and even homologically unital, by the chapter 3) amounts to the study of unital differential graded categories. Concerning differential graded categories and their derived categories, we refer to [Kel94a], [Kel99].

Chapter 8 : The A_∞-category of A_∞-functors.

Let $\mathcal{A}$ and $\mathcal{B}$ be two strictly unital A_∞-categories. We define (8.1.1 and 8.1.3) an A_∞-category $\mathsf{Func}_\infty(\mathcal{A}, \mathcal{B})$ whose objects are strictly unital A_∞-functors $\mathcal{A} \to \mathcal{B}$. The difficulty consists in defining the higher compositions of the morphisms between A_∞-functors. For this, we will use the torsion method from chapter 6. This A_∞-category is functorial in $\mathcal{A}$ and $\mathcal{B}$ (8.1.2). We deduce a 2-category cat_∞ whose objects are the the small strictly unital A_∞-categories and the morphism spaces $\mathcal{A} \to \mathcal{B}$ are the categories

$$\mathsf{cat}_\infty(\mathcal{A}, \mathcal{B}) = H^0\mathsf{Func}_\infty(\mathcal{A}, \mathcal{B}), \quad \mathcal{A}, \mathcal{B} \in \mathsf{Obj}\,\mathsf{cat}_\infty.$$

We characterize (8.2.2.3) then the elements

$$H \in \mathsf{Hom}_{\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})}(f, g), \quad f, g : \mathcal{A} \to \mathcal{B}$$

which become isomorphisms $f \to g$ in the category $\mathsf{cat}_\infty(\mathcal{A}, \mathcal{B})$. The proof of this characterization will use the existence of a *generalized* A_∞*-Yoneda functor* (8.2.1)

$$z : \mathsf{Func}_\infty(\mathcal{A}, \mathcal{B}) \to \mathcal{C}_\infty(\mathcal{A}, \mathcal{B})$$

which induces a quasi-isomorphism in the morphism spaces.

The A_∞-category $\mathsf{Func}_\infty(\mathcal{A}, \mathcal{B})$ was constructed independently independently by K. Fukaya [Fuk01b], V. Lyubashenko [Lyu02] and M. Kontsevich and Y. Soibelman [KS02a], [KS02b]. Although obtained by different methods, the compositions of $\mathsf{Func}_\infty(\mathcal{A}, \mathcal{B})$ of [Lyu02] are the same as ours.

Chapter 9 : A_∞-equivalences.

Let $\mathcal{A}$ be a strictly unital A_∞-category. In (9.1), we raise the notion of an isomorphism from $H^0\mathcal{A}$ to $\mathcal{A}$. We then show that an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is an A_∞-equivalence if and only if f_1 is a quasi-isomorphism and if it induces an equivalence of categories (in the classical sense) between $H^0\mathcal{A}$ and $H^0\mathcal{B}$ (9.2). Other proofs of this characterization (announced in [Kon98]) can be found in [Fuk01b] and [Lyu02].

Chapter 1

Homotopy Theory of A_∞-algebras

Introduction

Let us recall three classical results on A_∞-algebras:

1. *(Homotopy relation)* The relation of homotopy on A_∞-morphisms is an equivalence relation (1.3.1.3 *a*).
2. *(A_∞-quasi-isomorphism)* Any A_∞-quasi-isomorphism of A_∞-algebras is invertible up to homotopy (1.3.1.3 *b*).
3. *(Minimal model)* Every A_∞-algebra admits a minimal model (1.4.1.4).

In the literature, the results *1* and *2* are proved for A_∞-algebras satisfying certain conditions on their grading or their connectedness (see the references appearing in the body of the chapter). The goal of this chapter is to generalize them to any A_∞-algebras.

Chapter Plan

The chapter is divided into four sections. In section 1.1, we fix notations and we define free algebras and tensor coalgebras.

In section 1.2, we define A_∞-algebras, A_∞-morphisms and homotopies between A_∞-morphisms. We recall the bar and cobar constructions (1.2.2).

In section 1.3, we show the main result (1.3.1.2) of this chapter:

The category Cogc *of cocomplete differential graded coalgebras admits a model category structure which makes it Quillen-equivalent to the model category* Alg *of differential graded algebras. All objects of* Cogc *are cofibrant and the fibrant objects of* Cogc *are those which, as graded coalgebras, are isomorphic to reduced tensor coalgebras.*

The proof of the fact that the category Cogc admits such a structure was inspired by the work of V. Hinich [Hin01]. We consider filtered objects and study in this framework the properties of the bar and cobar constructions. The characterization of cofibrant objects will be immediate because cofibrations are injections. The characterization of fibrant objects will be a deeper result, a consequence of theorem (1.3.3.1): *the category of* A_∞*-algebras* Alg_∞ *admits a structure of "model category without limits"whose class of weak equivalences is formed by* A_∞*-quasi-isomorphisms.*

Our proof of this result will be entirely based on obstruction theory (see appendix B.1). It can therefore be interpreted as a consequence of the fact that the operad of A_∞-algebras is a cofibrant minimal model in the sense of M. Markl [Mar96] for the operad of associative algebras.

The A_∞-algebras are identified by the bar construction with the fibrant and cofibrant objects of Cogc. The results *1* and *2* cited above will then appear as special cases of fundamental results of Quillen's homotopic algebra (see appendix A).

In the section 1.4, we show (1.4.1.4) result *3* (minimal model). Our proof will use obstruction theory. Then, we compare (1.4.2.1) a minimal model obtained in this way with one obtained due to the perturbation lemma (see for example [HK91]).

1.1 Reminders and notations

1.1.1 Differential Graded objects

We fix notations that we will use throughout this chapter.

The basic category

Let $\mathbb{K}$ be a field. Let C be an abelian $\mathbb{K}$-linear, semi-simple, cocomplete category with exact filtered colimits (i.e. a semi-simple Grothendieck $\mathbb{K}$-category). We further assume that C is endowed with the structure of a $\mathbb{K}$-bilinear monoidal category given by the functor

$$\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C},$$

a unit object e, and associativity and unitality constraints

$$X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z, \qquad X \otimes e \simeq X \simeq e \otimes X, \qquad X, Y, Z \in \mathsf{C}.$$

We suppose that for all objects X of C, the functors $X \otimes -$ and $- \otimes X$ are exact and commute with filtered colimits.

The category of $\mathbb{K}$-vector spaces satisfies the above hypotheses. The reason why we work in a more general framework is the natural appearance of other examples in the study of A_∞-categories (see chapter 5).

Graded objects

A *graded object* (in C) is a sequence $M = (M^p)_{p \in \mathbf{Z}}$ of objects in C. Let M and L be two graded objects The *category* $\mathcal{G}r\mathsf{C}$ of graded objects has for the space of morphisms from M to L the $\mathbf{Z}$-graded vector space with components

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(M, L)^r = \prod_p \mathsf{Hom}_{\mathsf{C}}(M^p, L^{p+r}), \qquad r \in \mathbf{Z}.$$

We call the elements of the r-th component *graded morphisms of degree* r. The *tensor product* of two graded objects M and L has for components

$$(M \otimes L)^n = \bigoplus_{p+q=n} M^p \otimes L^q, \qquad n \in \mathbf{Z}.$$

Let $f : M \to M'$ and $g : L \to L'$ be two graded morphisms of r and s. The *tensor product*

$$f \otimes g : M \otimes L \to M' \otimes L'$$

is a morphism of degree $r+s$ whose n-th component is induced by the morphisms

$$(-1)^{ps} f^p \otimes g^q : M^p \otimes L^q \to M'^{p+r} \otimes L'^{q+s}, \qquad p+q=n.$$

The unit object for the graded tensor product is the graded object of which all the components are zero, except the 0-th, which is e. We also denote this by e. The category $\mathcal{G}r\mathsf{C}$ is thus equipped with the structure of a monoidal category. We define the *suspension functor* $S : \mathcal{G}r\mathsf{C} \to \mathcal{G}r\mathsf{C}$ by

$$(SM)^i = M^{i+1}, \qquad i \in \mathbf{Z}.$$

We denote

$$s_M : M \to SM$$

the *graded functorial morphism* of degree -1 with components

$$s_M^i = \mathbf{1}_{M^i} : M^i \to (SM)^{i-1}, \qquad i \in \mathbf{Z}.$$

The morphism s^{-1} is denoted by ω. Note the equality

$$\omega^{\otimes i} \circ s^{\otimes i} = (-1)^{\frac{i(i-1)}{2}} \mathbf{1}_{M^{\otimes i}}.$$

Differential graded objects

A *differential graded object* (or *complex*) is a pair (M,d), where M is a graded object and d is a *differential*, that is, an endomorphism of M of degree $+1$, such that $d^2 = 0$. The *subobject* $Z^iM = \ker d^i$ of M^i is the object of *cycles* of degree i of the complex M. The *subobject* $B^iM = \operatorname{Im} d^{i-1}$ of Z^iM is the object of *boundaries* of degree i in the complex M. Let (M, d_M) and (L, d_L) be two complexes, the space of graded morphisms $\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(M,L)$ with the differential δ given componentwise by

$$\begin{array}{rccl} \delta^r : & \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(M,L)^r & \to & \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(M,L)^{r+1}, \qquad r \in \mathbf{Z}. \\ & f & \mapsto & d_L \circ f - (-1)^r f \circ d_M \end{array}$$

The *category* $\mathcal{C}\mathsf{C}$ has for objects complexes and for morphism spaces

$$\mathsf{Hom}_{\mathcal{C}\mathsf{C}}(M,L) = Z^0(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(M,L), \delta).$$

If M and L are two complexes, we equip the graded tensor product $M \otimes L$ with the differential

$$d_{M\otimes L} = d_M \otimes \mathbf{1}_L + \mathbf{1}_M \otimes d_L.$$

We have thus equipped $\mathcal{C}\mathsf{C}$ with the structure of a monoidal category with a graded unit object e with zero differential. If M is a complex, we endow its suspension SM with the differential

$$d_{SM} = -s_M \circ d_M \circ s_M^{-1}.$$

The functor *homology* $H : \mathcal{C}\mathsf{C} \to \mathcal{G}r\mathsf{C}$ sends a complex M to the graded object HM with components

$$H^iM = Z^iM/B^iM, \qquad i \in \mathbf{Z}.$$

A *quasi-isomorphism* of $\mathcal{C}\mathsf{C}$ is a morphism which induces an isomorphism on homology. A complex is *acyclic* if it is quasi-isomorphic to the zero object. Two morphisms of complexes $f, g : M \to L$ are *homotopic* if there exists a morphism $r : M \to L$ of degree -1 such that $\delta(r) = f - g$. Homotopy is an equivalence relation. The *category* $\mathcal{H}\mathsf{C}$ has for objects complexes and for the space of morphisms from M to L homotopy classes of morphisms in the category $\mathcal{C}\mathsf{C}$:

$$\mathsf{Hom}_{\mathcal{H}\mathsf{C}}(M, L) = H^0(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(M, L), \delta).$$

We denote (again) by $H : \mathcal{H}\mathsf{C} \to \mathcal{G}r\mathsf{C}$ the functor induced by the homology functor.

1.1.2 Algebras and coalgebras

Algebras

Let M be one of the categories C, $\mathcal{G}r\mathsf{C}$ or $\mathcal{C}\mathsf{C}$. An *algebra* (A, μ) in M is an object A equipped with an associative multiplication $\mu : A \otimes A \to A$ (and of degree 0 if $\mathsf{M} = \mathcal{G}r\mathsf{C}$). Define $\mu^{(2)} = \mu$, and for all $n \geq 3$, $\mu^{(n)} : A^{\otimes n} \to A$ by

$$\mu^{(n)} = \mu(\mathbf{1} \otimes \mu^{(n-1)}).$$

For $n \geq 1$, we call $\mathsf{cok}\,\mu^{(n+1)}$ the *algebra of n-irreducibles* of A.

Let $f, g : A \to B$ be two morphisms of algebras. An *(f,g)-derivation* is a morphism $D : A \to B$ satisfying the Leibniz rule

$$D \circ \mu = \mu \circ (f \otimes D + D \otimes g).$$

A *derivation* for an algebra A is a $(\mathbf{1}_A, \mathbf{1}_A)$-derivation.

Let V be a graded in M. The *reduced tensor algebra over V* is the object

$$\overline{T}V = \bigoplus_{i \geq 1} V^{\otimes i}$$

endowed with the multiplication μ whose components

$$V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes i+j} \to \overline{T}V$$

are given by the associativity constraint of the monoidal category M. An algebra A of M is *free* if it is isomorphic to $\overline{T}V$ for an object V in M. We thus have $V \simeq \mathsf{cok}\,\mu_A$.

Lemma 1.1.2.1 (Universal property of tensor algebra)**.**
Let (A, μ) be an algebra. For $n \geq 1$, we by denote $j_n : V^{\otimes n} \to \overline{T}(V)$ the canonical injection.

a. The map $f \mapsto f \circ j_1$ is a bijection from the set of algebra morphisms $\overline{T}(V) \to A$ to the set of morphisms $V \to A$ in M (of degree 0 if $\mathsf{M} = \mathcal{G}r\mathsf{C}$). The inverse map associates to $g : V \to A$ the algebra morphism $\mathsf{mor}(g) : \overline{T}V \to A$ whose n-th component is

$$V^{\otimes n} \xrightarrow{g^{\otimes n}} A^{\otimes n} \xrightarrow{\mu^{(n)}} A, \qquad n \geq 1.$$

b. Let $f, g : A \to B$ be two algebra morphisms. The map $D \mapsto D \circ j_1$ is a bijection from the set of (f,g)-derivations to the set of morphisms $V \to A$ in M. The inverse map associates to $h : V \to A$ the (f,g)-derivation $\mathsf{der}(h) : \overline{T}V \to A$ whose n-th component is

$$\mu^{(n)} \circ \left(\sum_{l+1+j=n} (f^{\otimes l} \otimes h \otimes g^{\otimes j}) \right), \qquad n \geq 1.$$

□

A *graded algebra* (resp. *differential graded algebra*) is an algebra in the category $\mathcal{G}r\mathsf{C}$ (resp. the category $\mathcal{C}\mathsf{C}$). We denote by Alg the category of differential graded algebras. A morphism in Alg is a *quasi-isomorphism* if it induces an isomorphism on homology. A differential graded algebra is *almost free* if it is free as a graded algebra. Two morphisms $f, g : A \to B$ of Alg are *homotopic* if there exists a (f, g)-derivation $H : A \to B$ graded of degree -1 such that

$$f - g = dH + Hd.$$

It will follow from proposition A.13 applied to example 1.3.1.1 that, if the algebra A is almost free, the homotopy relation is an equivalence relation on the set of morphisms of algebras from A to B.

Coalgebras

A *coalgebra* in M is an object C equipped with a coassociative comultiplication $\Delta : C \to C \otimes C$, i.e. $(\Delta \otimes \mathbf{1})\Delta = (\mathbf{1} \otimes \Delta)\Delta$. Define $\Delta^{(2)} = \Delta$ and, for any $n \geq 3$, $\Delta^{(n)} : C \to C^{\otimes n}$ by

$$\Delta^{(n)} = (\mathbf{1}^{\otimes n-2} \otimes \Delta) \circ \Delta^{(n-1)}.$$

Let $n \geq 1$. The kernel $C_{[n]} = \ker \Delta^{(n+1)}$ is a sub-coalgebra of C; we call it the *sub-coalgebra of n-primitives* of C. The increasing sequence of sub-coalgebras

$$C_{[1]} \subset C_{[2]} \subset C_{[3]} \subset \cdots$$

is the *primitive filtration* of a coalgebra C. A coalgebra C is *cocomplete* if

$$\operatorname{colim} C_{[i]} = C.$$

Let f and $g : C \to B$ be two morphisms of coalgebras. A (f, g)-*coderivation* is a morphism $D : C \to B$ satisfying the dual Leibniz rule

$$\Delta \circ D = (f \otimes D + D \otimes g) \circ \Delta.$$

A *coderivation* of C is a $(\mathbf{1}_C, \mathbf{1}_C)$-coderivation.

Let V be an object in M. A *reduced tensor coalgebra over V* is an object

$$\overline{T^c}V = \bigoplus_{i \geq 1} V^{\otimes i}$$

endowed with a comultiplication whose n-th component

$$V^{\otimes n} \longrightarrow \oplus_{i+j=n} V^{\otimes i} \otimes V^{\otimes j} \longrightarrow \overline{T^c}V \otimes \overline{T^c}V,$$

is the sum of the morphisms $V^{\otimes n} \to V^{\otimes i} \otimes V^{\otimes j}$ given by the associativity constraint from the monoidal structure of M. Note that if C is isomorphic to a reduced tensor coalgebra, it is isomorphic to $\overline{T^c}(C_{[1]})$. Reduced tensor coalgebras are cocomplete.

Lemma 1.1.2.2 (Universal property of tensor coalgebras)**.**
Let C be a cocomplete coalgebra. For $n \geq 1$, we denote by $p_n : \overline{T^c}(V) \to V^{\otimes n}$ the canonical projection

a. The map $f \mapsto p_1 \circ f$ is a bijection from the set of morphisms of coalgebras to the set of morphisms $C \to V$ in M (of degree 0 if $\mathsf{M} = \mathcal{G}r\mathsf{C}$). The inverse map associates to $g : C \to V$ a coalgebra morphism $\mathsf{mor}(g) : C \to \overline{T^c}V$ where the n-th component is

$$C \xrightarrow{\Delta^{(n)}} C^{\otimes n} \xrightarrow{g^{\otimes n}} V^{\otimes n}, \qquad n \geq 1.$$

b. Let $f, g : C \to \overline{T^c}V$ be two coalgebra morphisms. The map $D \mapsto p_1 \circ D$ is a bijection from the set of (f,g)-coderivations $C \to \overline{T^c}V$ to the set of morphisms $C \to V$. The inverse map associates to $h : C \to V$ a (f,g)-coderivation $\mathsf{cod}(h) : C \to \overline{T^c}V$ where the n-th component is

$$\left(\sum_{l+1+j=n} (f^{\otimes l} \otimes h \otimes g^{\otimes j}) \right) \circ \Delta^{(n)}, \qquad n \geq 1.$$

□

Remark 1.1.2.3. The canonical isomorphism

$$e \xrightarrow{\sim} e \otimes e$$

makes $C = e$ a coalgebra. It is not cocomplete. There is no nonzero morphism $C \to V$ that lifts to a morphism of coalgebras $C \to \overline{T^c}V$.

We denote by Cog the category of differential graded coalgebras and by Cogc the subcategory of Cog formed by the cocomplete coalgebras. Two morphisms $f, g : C \to B$ of differential graded coalgebras are *homotopic* if there exists a graded (f,g)-coderivation $H : C \to B$ of degree -1 such that

$$f - g = dH + Hd.$$

It will follow from Theorem 1.3.1.2 and Lemma A.12 that, if the graded coalgebra underlying B is isomorphic to a reduced graded tensor coalgebra, then homotopy is an equivalence relation on the set of coalgebra morphisms from C to B.

1.2 A_∞-algebras and A_∞-coalgebras

1.2.1 Definitions

Definition 1.2.1.1. Let n be an integer ≥ 1. An A_n-*algebra* is an object A of $\mathcal{G}r\mathsf{C}$ equipped with a family of graded morphisms

$$m_i : A^{\otimes i} \to A, \qquad 1 \leq i \leq n,$$

of degree $2 - i$ satisfying, for all $1 \leq m \leq n$, the equation

$$(*_m) \qquad \sum (-1)^{jk+l} m_i (\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = 0$$

in $\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes m}, A)$, where the integers i, j, k, l are such that $j + k + l = m$ and $i = j + 1 + l$. An A_∞-*algebra* (or *strongly homotopy associative algebra*) is an object A in $\mathcal{G}r\mathsf{C}$ equipped with graded morphisms $m_i : A^{\otimes i} \to A$, $i \geq 1$, of degree $2 - i$ satisfying the equation $(*_m)$ for all $m \geq 1$.

Definition 1.2.1.2. An $\mathrm{A_n}$-*morphism* of $\mathrm{A_n}$-algebras $f : A \to B$ is a family of graded morphisms

$$f_i : A^{\otimes i} \to B, \qquad 1 \leq i \leq n,$$

of degree $1-i$ satisfying, for all $1 \leq m \leq n$, the equation

$$(**_m) \qquad \sum (-1)^{jk+l} f_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = \sum (-1)^s m_r(f_{i_1} \otimes \ldots \otimes f_{i_r})$$

in $\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes m}, B)$, where the integers i, j, k, l in the left sum are such that $j+k+l = m$ and $i = j+1+l$ and

$$s = \sum_{2 \leq u \leq r} \Big((1-i_u) \sum_{1 \leq v \leq u} i_v \Big).$$

An $\mathrm{A_n}$-morphism f is *strict* if $f_i = 0$ for all $i \geq 2$. A *composition* of an $\mathrm{A_n}$-morphism $f : A \to B$ with an $\mathrm{A_n}$-morphism $g : B \to C$ is defined by

$$(gf)_m = \sum_r \sum_{i_1+..+i_r=m} (-1)^s g_r(f_{i_1} \otimes \ldots \otimes f_{i_r})$$

as a morphism from $A^{\otimes m}$ to C, where s is the same sign as before. The *identity* of an $\mathrm{A_n}$-algebra A is the $\mathrm{A_n}$-morphism such that $f_1 = \mathbf{1}_A$ and $f_i = 0$ if $2 \leq i \leq m$. An A_∞-*morphism* is a family of graded morphisms $f_i : A^{\otimes i} \to B$, $i \geq 1$, of degree $1-i$ satisfying the equation $(**_m)$ for all $m \geq 1$. For the A_∞-algebras, the components of the composition and identity are defined by the same formulas as for the $\mathrm{A_n}$-algebras.

It will result from section 1.2.2 that we thus obtain a category.

Denote by Alg_∞ the *category of* A_∞*-algebras*. Likewise, for all $n \geq 1$, we obtain a *category* Alg_n *of* $\mathrm{A_n}$*-algebras*.

Remark 1.2.1.3. The definition of A_∞-algebras implies the following formulas which explain the other name of an A_∞-algebra: strongly homotopically associative. The equality

$$(*_1) \qquad m_1\, m_1 = 0$$

shows that (A, m_1) is a complex. The equality

$$(*_2) \qquad m_1\, m_2 = m_2\, (m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$$

of morphisms $A^{\otimes 2} \to A$ means that the differential m_1 is a derivation for the multiplication m_2. The equality

$$(*_3) \qquad m_2(m_2 \otimes \mathbf{1} - \mathbf{1} \otimes m_2) = m_1 m_3 + m_3\, (m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1)$$

of morphisms $A^{\otimes 3} \to A$ expresses that the lack of associativity of m_2 is equal to the boundary of m_3 in the complex

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((A, m_1)^{\otimes 3}, (A, m_1)).$$

This means that the graded object A endowed with the multiplication m_2 is an algebra whose multiplication is associative up to homotopy.

Similarly, the definition of an A$_\infty$-morphism $f : A \to B$ implies the following formulas. The equality

$$(**_1) \qquad f_1 m_1 = m_1 f_1$$

shows that the graded morphism f_1 is a morphism of complexes. The equality

$$(**_2) \qquad f_1 m_2 = m_2 (f_1 \otimes f_1) + m_1 f_2 + f_2 (m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$$

means that the lack of compatibility of f_1 with the multiplications of A and B are measured by the boundary of f_2 in

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((A, m_1)^{\otimes 2}, (B, m_1)).$$

Remark 1.2.1.4. If (V, d) is a complex, the morphisms

$$m_1 = d, \qquad m_i = 0 \quad \text{for} \quad \mathrm{i} \geq 2$$

define an A$_\infty$-algebra structure on V. The category $\mathcal{C}\mathsf{C}$ of complex is a non-full subcategory of Alg_∞.

Remark 1.2.1.5. If $((A, d), m)$ is a differential graded algebra, the morphisms

$$m_1 = d, \qquad m_2 = m, \qquad m_i = 0 \quad \text{for} \quad \mathrm{i} \geq 3$$

define an A$_\infty$-algebra structure on A. Conversely, if in an A$_\infty$-algebra A, the multiplications m_i are zero for $i \geq 3$, the complex (A, m_1) equipped with multiplication $m_2 : A \otimes A \to A$ is a differential graded algebra. The category Alg of differential graded algebras is a non-full subcategory of Alg_∞.

Definition 1.2.1.6. An A$_\infty$*-quasi-isomorphism* f is an A$_\infty$-morphism such that f_1 is a quasi-isomorphism of complexes.

Definition 1.2.1.7. Let A and A' be two A$_\infty$-algebras. Let f and g be two A$_\infty$-morphisms $A \to A'$. A *homotopy between* f *and* g is a family of morphisms

$$h_i : A^{\otimes i} \to B, \qquad 1 \leq i,$$

of degree $-i$ satisfying, for all $1 \leq n$, the equation $(***_n)$

$$\begin{aligned} f_n - g_n \quad = \quad & \sum (-1)^s m_{r+1+t} (f_{i_1} \otimes \ldots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \ldots \otimes g_{j_t}) \\ & + \sum (-1)^{jk+l} h_i (\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) \end{aligned}$$

in $\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes n}, B)$, where the sum of the integers $i_1, \ldots i_r, k, j_1, \ldots j_t$ is n, where $j + k + l = n$ and where

$$s = t + \sum_{1 \leq \alpha \leq t} (1 - j_\alpha)(n - \sum_{u \geq \alpha} j_u) + k \sum_{1 \leq u \leq r} i_u + \sum_{2 \leq \alpha \leq r} (1 - i_\alpha) \sum_{u < \alpha} i_u.$$

Two A$_\infty$-morphisms of A$_\infty$-algebras f and g are *homotopic* if there exists a homotopy between f and g.

Definition 1.2.1.8. A A$_\infty$*-coalgebra* (or *strongly homotopically co-associative coalgebra*) is an object C of $\mathcal{G}r\mathsf{C}$ endowed with a family of graded morphisms

$$\Delta_i : C \to C^{\otimes i}, \qquad i \geq 1,$$

of degree $2-i$ such that the morphism

$$S^{-1}C \longrightarrow \prod_{i\geq 1}(S^{-1}C)^{\otimes i}$$

whose components are

$$-\omega^{\otimes i}\circ\Delta_i\circ s \qquad (\text{where} \quad \omega = s^{-1})$$

is factorized by the monomorphism

$$\bigoplus_{i\geq 1}(S^{-1}C)^{\otimes i} \longrightarrow \prod_{i\geq 1}(S^{-1}C)^{\otimes i}$$

and that, for all $m\geq 1$, we have

$$\sum(-1)^{i+jk}(\mathbf{1}^{\otimes i}\otimes\Delta_j\otimes\mathbf{1}^{\otimes k})\Delta_l = 0,$$

where the integers i, j, k, l in the left sum are such that $i+j+k=m$ and $l=i+1+k$.

The cobar construction below will help to better understand this definition.

1.2.2 Bar and cobar constructions

The bar construction is due to S. Eilenberg and S. Mac Lane [EML53] for differential graded algebras (see also [Car55]) and due to J. Stasheff [Sta63b] for A_∞-algebras. It allows, among other things, a reformulation of the definition of A_∞-structures. It also gives an explanation (Remark 1.2.2.2) of the signs appearing in the equations $(*_m)$ of the definition of A_∞-algebras. The cobar construction is analogous to the bar construction in the case of A_∞-coalgebras [Ada56].

Bar construction

Let A be a graded object. Let a family of graded morphisms of

$$m_i : A^{\otimes i}\to A, \qquad i\geq 1,$$

of degree $2-i$. For all $i\geq 1$, we define a bijection

$$\begin{array}{ccc}\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes i}, A) & \to & \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((SA)^{\otimes i}, SA)\\ m_i & \mapsto & b_i\end{array}$$

by the relation

$$b_i = -s\circ m_i\circ\omega^{\otimes i} \qquad \text{where} \qquad \omega = s^{-1}.$$

Note that the morphism b_i has degree $+1$.

Let $\overline{T^c}(SA)$ be the reduced graded tensor coalgebra on SA. By virtue of Lemma 1.1.2.2, the morphism

$$\bigoplus_{i\geq 1}(SA)^{\otimes i}\to SA$$

with components b_i can be lifted into a single coderivation

$$b : \overline{T^c}(SA) \longrightarrow \overline{T^c}(SA).$$

Lemma 1.2.2.1 (J. Stasheff [Sta63b])**.** The following propositions are equivalent:

a. The m_i define an A_∞-algebra structure on A.

b. For each $m \geq 1$, the following equation holds

$$\sum_{\substack{j+k+l=m\\ j+1+l=i}} b_i(\mathbf{1}^{\otimes j} \otimes b_k \otimes \mathbf{1}^{\otimes l}) = 0.$$

c. The coderivation b is a differential, i.e. $b^2 = 0$.

Proof. The equivalence between the first two points is the result of the following equalities in $\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes i}, SA)$

$$\begin{aligned} b_i \circ (\mathbf{1}^{\otimes j} \otimes b_k \otimes \mathbf{1}^{\otimes l}) &= sm_i\omega^{\otimes i} \circ (\mathbf{1}^{\otimes j} \otimes sm_k\omega^{\otimes k} \otimes \mathbf{1}^{\otimes l}) \\ &= (-1)^l \ sm_i \circ (\omega^{\otimes j} \otimes (m_k \circ \omega^{\otimes k}) \otimes \omega^{\otimes l}) \\ &= (-1)^{l+jk} \ sm_i \circ (\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) \circ \omega^{\otimes n}. \end{aligned}$$

As the coderivation b is of odd degree, its square is still a coderivation. By Lemma 1.1.2.2, we therefore have $b^2 = 0$ if and only if $p_1 b^2 = 0$. This shows the equivalence of the last two points. ☐

Remark 1.2.2.2 (Signs)**.** The choice of the bijection $m_i \leftrightarrow b_i$ is not canonical. Another choice would give other signs in the equations $(*_m)$ of the definition 1.2.1.1.

Definition 1.2.2.3. The differential graded coalgebra $(\overline{T^c}(SA), b)$ associated to an A_∞-algebra A is denoted BA and is called the *bar construction* of A.

Let A and A' be two A_∞-algebras. For all $i \geq 1$, we define a bijection

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes i}, A') \quad \to \quad \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((SA)^{\otimes i}, SA')$$

by the relation

$$\omega \circ F_i = (-1)^{|F_i|} f_i \circ \omega^{\otimes i}$$

where F_i is a graded morphism of degree $|F_i|$. Let there be a family of graded morphisms

$$f_i : A^{\otimes i} \to A', \qquad i \geq 1,$$

of degree $1 - i$. Let

$$F : BA \longrightarrow BA'$$

be the morphism of graded coalgebras which lifts the morphism

$$\bigoplus_{i \geq 1} (SA)^{\otimes i} \longrightarrow SA'$$

with components F_i.

A proof similar to that of lemma 1.2.2.1 shows that the f_i define a morphism of A_∞-algebras if and only if F is compatible with differentials. Thus, the equations $(**_m)$ are the translation of the fact that the (F, F)-coderivation $F \circ b_{BA} - b_{BA'} \circ F$ vanishes.

Let A and A' be two A_∞-algebras. Let f and g be two A_∞-morphisms of A_∞-algebras. Let F and G be coalgebra morphisms $BA \to BA'$ corresponding to f and g. Let $H : BA \to BA'$ be a (F, G)-coderivation of degree -1. It is determined (Lemma 1.1.2.2) by its composition with the projection on SA'

$$p_1 \circ H : BA \to SA'.$$

whose components are denoted

$$H_i : (SA)^{\otimes i} \to SA', \quad i \geq 1.$$

The bijections $F_i \leftrightarrow f_i$ send the morphisms H_i to the morphisms $h_i : A^{\otimes i} \to A'$, $i \geq 1$. This defines a bijection from the set of (F, G)-coderivations of degree -1 to the product of spaces of graded morphisms $A^{\otimes i} \to A'$, $i \geq 1$, of degree $-i$. This bijection sends a homotopy $H : BA \to BA'$ between the coalgebra morphisms F and G to the homotopy between the A_∞-morphisms f and g defined by the family

$$h_i : A^{\otimes i} \to A', \quad i \geq 1.$$

The equations $(***_m)$ of the definition 1.2.1.7 are the translation of the equation $F - G = \delta(H)$.

We thus obtain a functor $B : \mathsf{Alg}_\infty \to \mathsf{Cogc}$ called the *bar construction* functor. It sends homotopic A_∞-morphisms to homotopic morphisms of coalgebras. The bar construction induces an equivalence between the category of A_∞-algebras and the full subcategory Cogc formed of differential graded coalgebras whose underlying graded coalgebra is isomorphic to a reduced tensor graded coalgebra.

Let V be a graded object and $n \geq 1$. The sub-coalgebra of n-primitives of $\overline{T^c}V$ has the underlying graded space

$$\bigoplus_{1 \leq i \leq n} V^{\otimes i}.$$

We denote this coalgebra by $\overline{T^c_{[n]}}V$. A reasoning analogous to that which we have just made for the A_∞-algebras makes it possible to construct a fully faithful functor

$$B_n : \mathsf{Alg}_n \to \mathsf{Cogc}$$

which sends a A_n-algebra A to the differential graded coalgebra $(\overline{T^c_{[n]}}(SA), b)$, where b is the differential constructed using the bijection $b_i \leftrightarrow m_i$.

Cobar construction

Let C be a graded object. For $i \geq 1$, define the bijection

$$\begin{array}{ccc} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(C, C^{\otimes i}) & \xrightarrow{\sim} & \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(S^{-1}C, (S^{-1}C)^{\otimes i}) \\ \Delta_i & \longmapsto & D_i \end{array}$$

by the relation

$$D_i = -\omega^{\otimes i} \circ \Delta_i \circ s.$$

Consider a family of graded morphisms

$$\Delta_i : C \to C^{\otimes i}, \qquad i \geq 1,$$

of degree $2-i$ such that the morphism

$$S^{-1}C \longrightarrow \prod_{i\geq 1}(S^{-1}C)^{\otimes i}$$

whose components are the D_i, $i \geq 1$, is factorized by the monomorphism

$$\bigoplus_{i\geq 1}(S^{-1}C)^{\otimes i} \longrightarrow \prod_{i\geq 1}(S^{-1}C)^{\otimes i}.$$

By way of Lemma 1.1.2.1, the graded morphism $S^{-1}C \to \overline{T}S^{-1}C$ thus obtained extends to a unique derivation of algebras of $\overline{T}S^{-1}C$. Using Lemma 1.1.2.1, we show that we have $D^2 = 0$ if and only if the Δ_i define a A_∞-coalgebra structure on C. Thus, the differentials of the algebra $\overline{T}S^{-1}C$ are in bijection with the A_∞-coalgebra structures on the graded object C.

Definition 1.2.2.4. We denote by ΩC the differential graded algebra $(\overline{T}S^{-1}C, D)$ associated to an A_∞-coalgebra C. It's called the *cobar construction* of C.

The category Cog_∞ of A_∞-coalgebras has as objects A_∞-coalgebras. We define its morphisms in such a way that the construction cobar

$$\Omega : \mathsf{Cog}_\infty \longrightarrow \mathsf{Alg}$$

becomes a fully faithful functor. The category of differential graded coalgebras is then identified with a subcategory (not full) of the category of A_∞-coalgebras.

We also note by B (resp. Ω) the restriction of the bar (resp. cobar) construction to differential graded algebras (resp. cocomplete coalgebras).

Lemma 1.2.2.5. The functor $\Omega : \mathsf{Cogc} \to \mathsf{Alg}$ is left adjoint to the functor $B : \mathsf{Alg} \to \mathsf{Cogc}$.

Proof. This lemma is well known. Let A be an algebra and C a cocomplete coalgebra. It suffices to show that we have a functorial isomorphism

$$\mathsf{Hom}_{\mathsf{Cogc}}(C, BA) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Alg}}(\Omega C, A).$$

Let $F : C \to BA$ be a coalgebra morphism. As BA is a reduced graded tensor coalgebra, the data of F is equivalent (Lemma 1.1.2.2) to that of

$$f = p_1 F : C \to SA.$$

Let $\tau = \omega \circ f$. The condition $d_{BA} \circ F - F \circ d_C = 0$ results in the fact that τ is a *twisting cochain*, i.e. that we have

$$d_A \circ \tau + \tau \circ d_C + m \circ \tau^{\otimes 2} \circ \Delta = 0.$$

The graded morphism $f' = \tau \circ s$ extends in a unique way (Lemma 1.1.2.1) to a morphism of algebras $F' : \Omega C \to A$ because ΩC is free on $S^{-1}C$ as a graded algebra. The compatibility of F' with the differential is equivalent to the fact that τ is a twisting cochain. $\square$

1.3 Cogc as a model category

Section Plan

This section is divided into 5 subsections.

In the first subsection (1.3.1), we recall [Hin97] the model category structure on the category Alg of differential graded algebras. We state the main theorem (1.3.1.2) and deduce the *theorem of* A_∞*-quasi-isomorphisms* (1.3.1.3. *a*) and the *homotopy theorem* (1.3.1.3. *b*).

In the second subsection 1.3.1, we show the main theorem (1.3.1.2). For the characterization of fibrant objects of Cogc, we will need some results from the next subsection.

In subsection 1.3.3, we show that the category Alg_∞ admits the structure of a "model category without limits" (1.3.3.1). We then show that the bar construction $B : \mathsf{Alg}_\infty \to \mathsf{Cogc}$ is compatible with the ("limitless") model category structures of Alg_∞ and Cogc (1.3.3.5).

In subsection 1.3.4, we compare left homotopy (in the sense of model categories) with homotopy "in the classical sense" on morphisms of cocomplete differential graded coalgebras.

In subsection 1.3.5, we compare weak equivalences of Cogc with quasi-isomorphisms of Cogc.

1.3.1 The principal theorem

The reader who is not familiar with model categories in the sense of Quillen will find in the appendix A some reminders of certain key statements and the classic references.

The model category Alg

In the category Alg of differential $\mathbf{Z}$-graded algebras (1.1.2), consider the following three classes of morphisms:

- the class Qis of quasi-isomorphisms,
- the class $\mathcal{F}ib$ of morphisms $f : A \to B$ such that f^n is an epimorphism for all $n \in \mathbf{Z}$,
- the class $\mathcal{C}of$ of morphisms which have the left lifting property with respect to the morphisms in $Qis \cap \mathcal{F}ib$.

Let E be one of the full subcategories of Alg whose objects are respectively

(I) the algebras A such that $A^p = 0$ for all $p > 0$,

(II) the algebras A such that $A^p = 0$ for all $p \leq 0$.

H. Munkholm proved in [Mun78] that E becomes a model category if it is equipped with $\mathsf{E} \cap Qis$, $\mathsf{E} \cap \mathcal{F}ib$, and the class of morphisms E which have the left lifting property respect to the morphisms of $\mathsf{E} \cap Qis \cap \mathcal{F}ib$. H. Munkholm's result was reinforced by V. Hinich:

Theorem 1.3.1.1 (Hinich [Hin97])**.** The category Alg endowed with the classes of morphisms defined above is a model category. Cofibrant algebras are algebras that are isomorphic to an almost free algebra. All algebras are fibrant. □

The more general case where the base ring is not a field is due to J. F. Jardine [Jar97]. S. Schwede and B. Shipley [SS00] generalized these results to categories of algebras over monoidal model categories.

The principal theorem and its consequences

In the category Cogc of differential graded cocomplete cogebras, we consider the following three classes of morphisms:

- the class $\mathcal{E}q$ of *weak equivalences* is formed of the morphisms $f : C \to D$ such that $\Omega F : \Omega C \to \Omega D$ is a quasi-isomorphism of algebras,
- the class $\mathcal{C}of$ of *cofibrations* is made up of the morphisms $f : C \to D$ which, as morphisms of complexes, are monomorphisms,
- the class $\mathcal{F}ib$ of *fibrations* is made up of morphisms which have the right-lifting-property with respect to trivial cofibrations.

It turns out that the class of weak equivalences is strictly included in the class of quasi-isomorphisms of coalgebras (see Section 1.3.5). On the other hand, it is well known (and we will prove it again, see the Proposition 1.3.5.1) that a quasi-isomorphism between cocomplete coalgebras is a weak equivalence if the two coalgebras are concentrated in degrees < -1 or in degrees ≥ 0.

Theorem 1.3.1.2.

a. The category Cogc equipped with the three classes of morphisms above is a model category. All its objects are cofibrant. An object of Cogc is fibrant if and only if its underlying graded coalgebra is isomorphic to a reduced tensor coalgebra.

b. Equip the category Alg with the model category structure of Theorem 1.3.1.1. The pair of adjoint functors (Ω, B) from Cogc to Alg is a Quillen equivalence.

Proof. See Section 1.3.2. □

Point b of the theorem reinforces classical theorems (see [Moo71], [HMS74, th. 4.4 and 4.5]). It seems to be new in the form we give. Our proof is an adaptation of Hinich's [Hin01], based in turn on Quillen's [Qui69]. The fact that the bar and cobar functors induce inverse equivalences of each other in the homotopy categories is non-trivial but its proof is not very difficult. Let us now deduce, using homotopy algebra techniques of Quillen (see appendix A) the A_∞-*quasi-isomorphism theorem*, the *homotopy theorem* and the generalization of theorem [Mun78, Thm. 6.2] of H. J. Munkholm.

Corollary 1.3.1.3.

a. The homotopy relation (see Definition 1.2.1.7) in Alg_∞ is an equivalence relation.

b. A quasi-isomorphism of A_∞-algebras is a homotopy equivalence (i.e. an isomorphism in the quotient category of Alg_∞ by the homotopy relation).

c. Let dash be the full subcategory of Alg_∞ consisting of differential graded algebras. Let $\sim$ denote the homotopy relation on dash. The inclusion $\mathsf{Alg} \hookrightarrow \mathsf{dash}$ induces an equivalence

$$\mathsf{Alg}[Qis^{-1}] \xrightarrow{\sim} \mathsf{dash}/\sim .$$

The idea of point c goes back to J. Stasheff and S. Halperin [SH70]. It was proved under the conditions **(I)** or **(II)** (see top of this section) by H. J. Munkholm [Mun78]. The points a and b have been known (especially among rational homotopy specialists) since the beginning of the 80s, at least for connected A_∞-algebras (i.e. concentrated in homological degrees ≥ 1), see for example A. Prouté [Pro85, chap. 4] or T. V. Kadeishvili [Kad87].

Proof. From Section 1.2.2, we know that two morphisms of A_∞-algebras

$$f, g : A \to A'$$

are homotopic if and only if Bf and Bg are homotopic morphisms of coalgebras. By the main theorem (1.3.1.2), the coalgebra BA' is fibrant in Cogc and every object of Cogc is cofibrant. Let us provisionally accept (see Proposition 1.3.4.1 below) the following result: the homotopy relation in the classical sense on $\mathsf{Hom}_{\mathsf{Cogc}}(BA, BA')$ is equal to the left homotopy relation for the model category Cogc.

a. This is Lemma A.12 applied to the closed model category Cogc.

b. A A_∞-quasi-isomorphism $f : A \to A'$ induces (see Proposition 1.3.3.5 below) a morphism

$$Bf : BA \to BA',$$

which is a weak equivalence of Cogc between fibrant and cofibrant objects. It is therefore invertible up to homotopy in Cogc (see Proposition A.13).

c. By the main theorem 1.3.1.2, the functor B induces an equivalence

$$\mathsf{Alg}[Qis^{-1}] = \mathsf{Ho}\,\mathsf{Alg} \xrightarrow{\sim} \mathsf{Ho}\,\mathsf{Cogc}.$$

We have the equivalence (see Proposition A.13)

$$\mathsf{Cogc}_{\mathsf{cf}}/\!\sim\; \xrightarrow{\sim} \mathsf{Ho}\,\mathsf{Cogc}.$$

The functor B takes its values from $\mathsf{Cogc}_{\mathsf{cf}}$. It therefore induces an equivalence

$$\mathsf{Alg}[Qis^{-1}] \xrightarrow{\sim} \mathsf{Cogc}_{\mathsf{cf}}/\!\sim.$$

Its image is isomorphic to $\mathsf{dash}/\!\sim$. □

1.3.2 Proof of the principal theorem

Our proof of the main theorem 1.3.1.2 requires the prior study of filtered algebras and cogebras.

Filtered Objects

Let M be one of the categories $\mathcal{G}r\mathsf{C}$ or $\mathcal{C}\mathsf{C}$. A *filtration of an object* X of M is an increasing sequence

$$X_0 \subset X_1 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots, \quad i \in \mathbf{N}$$

of subobjects of X. It is *exhaustive* if we have

$$\mathsf{colim}\, X_i = X.$$

It is *admissible* if it is exhaustive and if $X_0 = 0$. A *filtered object* of M is an object of M equipped with a filtration. Let X and Y be two filtered objects The *graded object* $\mathsf{Gr}X$ *associated to* X is defined by the sequence of objects of M

$$\mathsf{Gr}_0 = X_0, \quad \mathsf{Gr}_i X = X_i/X_{i-1}, \quad i \geq 1.$$

A morphism $f : X \to Y$ of M is a *morphism of filtered objects* if we have

$$f(X_i) \subset Y_i$$

for all $i \in \mathbf{N}$. The tensor product $X \otimes Y$ is endowed with the filtration defined below

$$(X \otimes Y)_i = \sum_{p+q=i} X_p \otimes Y_q, \quad i \in \mathbf{N}.$$

This endows the category of filtered objects of M with a monoidal category structure whose neutral element is the object e equipped with the filtration $e_i = e$, $i \in \mathbf{N}$. The suspension SX of the object of M underlying X is endowed with the filtration given by $(SM)_i = SM_i$, $i \in \mathbf{N}$.

A *filtered complex* is a filtered object in $\mathcal{C}\mathsf{C}$.

Definition 1.3.2.1. Let X and Y be two filtered complexes. A morphism $f : X \to Y$ is a *filtered quasi-isomorphism* if the morphisms

$$\mathsf{Gr}_i C \to \mathsf{Gr}_i D, \quad i \in \mathbf{N},$$

induced by f are quasi-isomorphisms of complexes.

A *filtered algebra* (resp. *filtered coalgebra*) is an algebra (resp. coalgebra) in the category of filtered complexes. A *admissible filtered coalgebra* is a C coalgebra endowed with an admissible filtration. Note that we then have

$$\Delta C_{i+1} \subset C_i \otimes C_i, \quad i \in \mathbf{N}.$$

We will show (Lemma 1.3.2.2) that any filtered quasi-isomorphism between admissible filtered coalgebras is a weak equivalence of Cogc.

Let C be a filtered cocommutative coalgebra, complete as a coalgebra. The filtration of C induces a filtration on each tensor power of $S^{-1}C$. We thus obtain an algebra filtration on the cobar construction ΩC. Let C and D be two filtered and cocomplete coalgebras. Let us equip the cobar constructions ΩC and ΩD with the filtrations induced by those of C and D. The cobar construction maps a morphism of filtered coalgebras $f : C \to D$ to a morphism of filtered algebras $\Omega f : \Omega C \to \Omega D$.

Let A be a filtered algebra. The filtration of A induces a coalgebra filtration on the bar construction BA of A. Let A and A' be two filtered algebras. Let us equip the bar constructions BA and BA' with the filtrations induced by those of A and A'. The bar construction maps a morphism of filtered algebras $f : A \to A'$ to a morphism of filtered coalgebras $Bf : BA \to BA'$.

Let C be a cocomplete coalgebra. The *primitive filtration* of the coalgebra C is defined by the sequence of sub-coalgebras of i-primitives $C_{[i]}$, for $i \geq 1$, completed by $C_{[0]} = 0$. Since the base category C is semi-simple, the primitive filtration of C is a coalgebra filtration. It is admissible and induces a filtration on ΩC, which in turn induces a filtration on the bar construction $B\Omega C$. We call this latter filtration the *C-primitive filtration* of $B\Omega C$.

Lemma 1.3.2.2. A filtered quasi-isomorphism of filtered admissible coalgebras is a weak equivalence.

Proof. Let C and D be two admissible coalgebras, and $f : C \to D$ be a filtered quasi-isomorphism. We will show that the algebra morphism

$$\Omega f : \Omega C \to \Omega D$$

is a filtered quasi-isomorphism for the filtrations of ΩC and ΩD induced by those of C and D. We recall that the differential of ΩC is the unique coderivation d that extends the morphism

$$S^{-1}C \to \Omega C$$

with non-zero components $\sigma d_C s$ and $\sigma \Delta s^{\otimes 2}$. Let's equip ΩC with the filtration induced by that of C. Let $i \geq 1$. As the filtration of C is admissible, $\mathsf{Gr}_i(C^{\otimes j}) = 0$ if $j > i$. Equip

$$\mathsf{Gr}_i \Omega C = \mathsf{Gr}_i\Big(\bigoplus_{1 \leq j \leq i} C^{\otimes j}\Big)$$

with the filtration

$$F_l = \mathsf{Gr}_i\Big(\bigoplus_{i-l \leq j \leq i} C^{\otimes j}\Big), \quad l \geq 0.$$

The contribution of $\omega \Delta s^{\otimes 2}$ in the differential d of $\mathsf{Gr}_i \Omega C$ decreases the filtration. Thus, only the morphism $\omega d_C s$ contributes to the differential of the graded object associated with F_l, for $l \geq 1$. The morphism

$$\mathsf{Gr}_i \Omega C \longrightarrow \mathsf{Gr}_i \Omega D$$

is filtered for this filtration, and it clearly induces a quasi-isomorphism in the graded objects. □

Lemma 1.3.2.3.

a. Let A and A' be two differential graded algebras. The bar construction maps a quasi-isomorphism of algebras $f : A \to A'$ to a filtered quasi-isomorphism $f : BA \to BA'$ fro the primitive filtration.

b. Let A be a differential graded algebra. The adjunction morphism

$$\phi : \Omega BA \longrightarrow A$$

is a quasi-isomorphism of algebras.

c. Let C be a cocomplete coalgebra. Equip C with the primitive filtration and $B\Omega C$ with the C-primitive filtration. The adjunction morphism

$$\psi : C \longrightarrow B\Omega C$$

is a filtered quasi-isomorphism.

Proof.
a. The primitive filtration of BA has the associated graded object

$$\mathsf{Gr}_i(BA) = (SA)^{\otimes i}, \quad i \in \mathbf{N}.$$

By the Künneth theorem, a quasi-isomorphism $f : A \to A'$ induces a quasi-isomorphism in these subquotients.

b. We are going to introduce exhaustive filtrations on both A and ΩBA in such a way that the adjunction morphism becomes a filtered quasi-isomorphism. Let's consider the filtration of A defined as $A_i = A$ for $i \geq 1$ and $A_0 = 0$. Now, let's equip ΩBA with the induced filtration from the primitive filtration of BA. The adjunction morphism

$$\phi : \Omega BA \longrightarrow A$$

is clearly a filtered morphism. It induces a morphism

$$\mathsf{Gr}_i(\Omega BA) \longrightarrow \mathsf{Gr}_i A, \quad i \in \mathbf{N},$$

in the graded objects, which is the identity of A if $i = 1$, and which is zero if $i \geq 2$. To show that the adjunction morphism is a quasi-isomorphism, it suffices to show that, for $i \geq 2$, the complex $\mathsf{Gr}_i(\Omega BA)$ is contractible Consider the complex $V = SA$. Notice that we have an isomorphism of complexes:

$$\bigoplus_{i\geq 1} \mathsf{Gr}_i(\Omega BA) \xrightarrow{\sim} \Omega\overline{T^c}V$$

which identifies the component $\mathsf{Gr}_i(\Omega BA)$, $i \geq 1$, with the sum of

$$S^{-1}V^{\otimes i_1} \otimes \ldots \otimes S^{-1}V^{\otimes i_k} \subset (S^{-1}\overline{T^c}V)^{\otimes k},$$

where $k \geq 1$ and where $i_1 + \ldots + i_k = i$. Let $i \geq 2$. Consider the graded morphism $r : \mathsf{Gr}_i(\Omega BA) \to \mathsf{Gr}_i(\Omega BA)$ of degree -1 given by the morphisms

$$S^{-1}V^{\otimes i_1} \otimes S^{-1}V^{\otimes i_2} \otimes \ldots \otimes S^{-1}V^{\otimes i_k} \to S^{-1}V^{\otimes i_1+i_2} \otimes \ldots \otimes S^{-1}V^{\otimes i_k}$$

which we define to be zero if $i_1 \neq 1$ and equivalent to $\eta \circ (s \otimes \mathbf{1}^{\otimes k})$ otherwise; here η is the natural isomorphism

$$V \otimes S^{-1}V^{\otimes i_2} \otimes \ldots \otimes S^{-1}V^{\otimes i_k} \xrightarrow{\sim} S^{-1}V^{\otimes 1+i_2} \otimes \ldots \otimes S^{-1}V^{\otimes i_k}.$$

We verify that the graded morphism r is a contracting homotopy of the complex $\mathsf{Gr}_i(\Omega BA)$.

c. We must demonstrate that the morphism of complexes

$$\psi : \mathsf{Gr}C \to \mathsf{Gr}(B\Omega C)$$

is a quasi-isomorphism. Let $W = \mathsf{Gr}(S^{-1}C)$. Since C is admissible, the comultiplication of $\mathsf{Gr}C$ is zero, and

$$\mathsf{Gr}(B\Omega C) \xrightarrow{\sim} B\Omega(\mathsf{Gr}C)$$

is the sum of complexes

$$V_i = \bigoplus SW^{\otimes i_1} \otimes \ldots \otimes SW^{\otimes i_k}, \quad i \geq 1,$$

where $k \geq 1$ and $i_1 + \ldots + i_k = i$. The composite of the morphism

$$\mathsf{Gr}C \to \mathsf{Gr}B\Omega C$$

with the projection onto V_i is zero if $i \geq 2$ and is the identity of $\mathsf{Gr}C$ if $i = 1$. It remains to show that V_i is contractible for $i \geq 2$. Let $i \geq 2$. Let $r : V_i \to V_i$ be a graded morphism of degree -1 defined by the morphisms

$$SW^{\otimes i_1+i_2} \otimes \ldots \otimes SW^{\otimes i_k} \longrightarrow SW^{\otimes i_1} \otimes SW^{\otimes i_2} \otimes \ldots \otimes SW^{\otimes i_k}$$

which are defined to be zero if $i_1 \neq 1$ and as $\eta \circ (s \otimes \mathbf{1}^{\otimes i-1})$ otherwise; here η is the natural isomorphism

$$S^{-1}W^{\otimes 1+i_2} \otimes \ldots \otimes S^{-1}W^{\otimes i_k} \xrightarrow{\sim} W \otimes S^{-1}W^{\otimes i_2} \otimes \ldots \otimes S^{-1}W^{\otimes i_k}.$$

We verify that the morphism r is a contracting homotopy of V_i. □

Proof of the main theorem 1.3.1.2

We start with some preliminary lemmas.

Lemma 1.3.2.4. Let C be a coalgebra and C' a sub-coalgebra of C such that $\Delta C \subset C' \otimes C'$. The cobar construction maps the inclusion $C' \hookrightarrow C$ to a standard cofibration (1.3.2.5).

To prove this lemma and the following one, we will need the following description from [Hin97] of cofibrations in Alg: Let $A^\sharp$ denote the underlying complex of a differential graded algebra A, and let $FV = \overline{T}V$ be the free differential graded algebra over the complex V. Consider a differential graded algebra A and a complex M. Let $\alpha : M \to A^\sharp$ be a morphism of complexes. We denote by $C(\alpha)$ the cone of α in the category $\mathcal{C}\mathsf{C}$. Let $A\langle M, \alpha\rangle$ be the colimit in Alg of the diagram

$$A \leftarrow F(A^\sharp) \to FC(\alpha).$$

Definition 1.3.2.5. A morphism $f : A \to B$ is a *standard cofibration* if it is the colimit of a sequence of composite morphisms

$$A = A_0 \to A_1 \to \ldots \to A_{n-1} \to A_n, \quad n \geq 1,$$

where all the arrows $A_i \to A_{i+1}$ are given by the canonical morphisms

$$A_i \to A_i\langle M_i, \alpha_i\rangle = A_{i+1}$$

for morphisms of complexes $\alpha_i : M_i \to A_i^\sharp$. A *trivial standard cofibration* is a standard cofibration such that all complexes M_i are contractible (i.e. isomorphic to 0 in $\mathcal{H}\mathsf{C}$.)

The following facts are proved in [Hin97]: Every cofibration is retracted from a standard cofibration. Similarly, any trivial cofibration is retracted from a trivial standard cofibration.

Proof of Lemma 1.3.2.4. Let E be the cokernel in the category of complexes of the inclusion $C' \hookrightarrow C$. Choose a section of $C \to E$ in the graded category to obtain an isomorphism

$$C' \oplus E \xrightarrow{\sim} C$$

of graded objects. As a graded algebra, the cobar construction $\Omega C = \Omega(C' \oplus E)$ is isomorphic to the coproduct of graded algebras

$$FS^{-1}C' \amalg FS^{-1}E,$$

where $F = \overline{T}$ as in (Section 1.3.1). The differential of ΩC is induced by the comultiplication of C and the differential of the complex C. According to the decomposition $C = C' \oplus E$, the comultiplication of C is given by two components

$$\Delta_{C'} : C' \to C' \otimes C' \quad \text{and} \quad \Delta_E : E \to C' \otimes C',$$

and the differential of C is given by the differential of C', that of E and a graded morphism $d : E \to C'$ of degree $+1$. Let the morphism of complexes

$$[D_1, D_2] : S^{-2}E \longrightarrow S^{-1}C' \oplus \left(S^{-1}C' \otimes S^{-1}C'\right)$$

whose components are defined by $s^{\otimes 2} \circ D_2 = \Delta_E \circ s^2$ and by $s \circ D_1 = d \circ s^2$. We denote

$$D : S^{-2}E \longrightarrow FS^{-1}C' \amalg FS^{-1}E$$

its composition with the injection of $S^{-1}C' \oplus \left(S^{-1}C' \otimes S^{-1}C'\right)$ into $FS^{-1}C' \amalg FS^{-1}E$. By construction, the differential graded algebra

$$\Omega C' \langle S^{-2}E, D\rangle$$

is the graded algebra $FS^{-1}C' \amalg FS^{-1}E$ whose differential is induced by the comultiplication of C', the differentials of the complexes C' and E, the morphism Δ_E and the morphism d. It is therefore isomorphic to ΩC as a differential graded algebra. □

Lemma 1.3.2.6.

a. The cobar construction preserves weak cofibrations and equivalences.

b. The bar construction preserves fibrations and weak equivalences.

Proof.
a. Let $i : C \rightarrowtail D$ be a cofibration of coalgebras. Consider the filtration of D defined by the sequence $D_i = i(C) + D_{[i]}$, $i \in \mathbf{N}$. Notice that D_0 is isomorphic to C, and for all $i \geq 1$, we have

$$\Delta(D_{i+1}) \subset D_i \otimes D_i.$$

Therefore, we can apply Lemma 1.3.2.4. It certifies that $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration. The morphism $\Omega C \to \Omega D$ is the countable composition of standard cofibrations $\Omega D_i \rightarrowtail \Omega D_{i+1}$. Hence, it is also a standard cofibration. The cobar construction preserves weak equivalences by the definition of weak equivalences in Cogc.

b. Let $p : A \twoheadrightarrow A'$ be a filtration of algebras. The morphism Bf is a fibration if it satisfies the right lifting property with respect to the trivial cofibrations $i : C \to D$ of coalgebras. Thanks[1] to the adjunction between the bar and cobar constructions, this property is equivalent to Ωi having the left lifting property with respect to p. But this is always true according to point *a*. Therefore, the morphism Bf is a fibration in Cogc.

Let $f : A \xrightarrow{\sim} A'$ be a quasi-isomorphism of algebras. We want to show that Bf is a weak equivalence, which means that ΩBf is a quasi-isomorphism. Thanks to point *b* of Lemma 1.3.2.3,

[1] Says "Grâce à l'adjonction"...

the vertical arrows in the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & A' \\
\Omega BA \arrow[u] \arrow[r] & \Omega BA' \arrow[u]
\end{tikzcd}$$

these are quasi-isomorphisms. By the saturation property of quasi-isomorphisms, the morphism ΩBf is also a quasi-isomorphism. □

Proof of point a *of Theorem 1.3.1.2*:

(CM1) The colimits of finite diagrams of coalgebras are determined by the colimits of the diagrams of the underlying complexes. The constructions of products and equalizers in the category of cocomplete coalgebras are dual to those of coproducts and coequalizers in the category of algebras, which are described in [Mun78, 3.3].

(CM2) This is a consequence of the definition of weak equivalences and axiom (CM2) for the model category structure on Alg.

(CM3) Cofibrations are stable under retracts because they are monomorphisms. Weak equivalences are also stable under retracts because the functor Ω sends a retract to a retract. As for fibrations, it's worth noting that a morphism p is a fibration if it has the right lifting property with respect to trivial cofibrations. It can be verified that a retract of such a morphism p also has the same lifting property.

(CM4) See (CM5).

(CM5) *factorization:*
Let $f : C \to D$ be a morphism in Cogc. By Axiom (CM5) for the model category structure on Alg, we have a factorization of Ωf as

$$\begin{tikzcd}
\Omega C \arrow[rr, "f"] \arrow[dr, "i"'] & & \Omega D \\
& A, \arrow[ur, "p"'] &
\end{tikzcd}$$

where the cofibration i (respectively, the fibration p) in Alg is a quasi-isomorphism. Thus, the morphism $B\Omega f : B\Omega C \to B\Omega D$ factors as $Bp \circ Bi$. Consider the following diagram in Cogc:

$$\begin{tikzcd}
& BA \prod_{B\Omega D} D \arrow[dr, "q"] \arrow[dd] & \\
C \arrow[rr, crossing over, "f"] \arrow[dd, "\in \mathcal{E}q"] & \arrow[dr, phantom, "cart."] & D \arrow[dd, "\in \mathcal{E}q"] \\
& BA \arrow[dr, "Bp"] & \\
B\Omega C \arrow[ur, "Bi"] \arrow[rr, "B\Omega f"'] & & B\Omega D.
\end{tikzcd}$$

Since the diagram is commutative, the morphism $f : C \to D$ is the composition

$$C \to B\Omega C \xrightarrow{Bi} BA$$

determining a morphism $\tilde{\imath} : C \to BA \prod_{B\Omega D} D$. We will show that

$$\begin{tikzcd} & BA \prod_{B\Omega D} D \arrow[dr, "q"] & \\ C \arrow[ur, "\tilde{\imath}"] \arrow[rr, "f"'] & & D \end{tikzcd}$$

furnishes a factorization of the morphism f in Cogc, where $\tilde{\imath}$ is a cofibration and q is a fibration. Next, we will demonstrate that the cofibration $\tilde{\imath}$ (res. the fibration q) is trivial.

According to point b of Lemma 1.3.2.6, the morphism Bp is a fibration in Cogc. The projection $q : BA \prod_{B\Omega D} D \to D$ is also a fibration because fibrations are stable under base change. Suppose for the moment that we know $BA \prod_{B\Omega D} D \to BA$ is a cofibration (See Lemma 1.3.2.7 below). The morphism $\tilde{\imath}$ is a monomorphism (i.e., a weak equivalence in Cogc) since the composition

$$C \to B\Omega C \xrightarrow{Bi} BA$$

as the composition is one. It remains to show that the cofibration $\tilde{\imath}$ (resp. the fibration q) is a weak equivalence Cogc. Suppose for the moment that we know $BA \prod_{B\Omega D} D \to BA$ is a weak equivalence (See Lemma 1.3.2.7 below). We know from point b of Lemma 1.3.2.6 that the morphism Bi (resp. Bp) is a weak equivalence. Since the morphism $C \to B\Omega C$ (resp. $D \to B\Omega D$) is a weak equivalence, $\tilde{\imath}$ (resp. q) is also a weak equivalence by the saturation property of the class of weak equivalences in Cogc.

(CM4) *lifting:*

a. Consider the commutative diagram in Cogc

$$\begin{tikzcd} E \arrow[r] \arrow[d, tail, "u"'] & C \arrow[d, two heads, "t"] \\ F \arrow[r] & D \end{tikzcd}$$

where t is a trivial fibration and u a cofibration. We are looking for a morphism α such that the two triangles in the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, tail, "u"'] & C \arrow[d, two heads, "t"] \\ F \arrow[r] \arrow[ur, dotted, "\alpha"] & D \end{tikzcd}$$

are commutative. Using the construction of the proof of (CM5), we factorize t into $q \circ \tilde{\imath}$, where the morphism $q : BA \prod_{B\Omega D} D \to D$ is a fibration and where the morphism $\tilde{\imath} : C \to BA \prod_{B\Omega D} D$ is a cofibration. By the saturation property of the class $\mathcal{E}q$, the morphisms $\tilde{\imath}$ and q are both weak equivalences.

The fibrations being morphisms having the right-lifting property with respect to trivial cofibrations, there exists a lift $r : BA\prod_{B\Omega D} D \to C$ in the diagram of Cogc

$$\begin{tikzcd} C \arrow[r, "\mathbf{1}"] \arrow[d, tail, "\tilde{i}"'] & C \arrow[d, two heads, "t"] \\ BA\prod_{B\Omega D} D \arrow[r, "q"'] & D. \end{tikzcd}$$

All we need to do is find a lifting in the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, tail, "u"'] & BA\prod_{B\Omega D} D \arrow[d, two heads, "q"] \\ F \arrow[r] \arrow[ur, dotted] & D, \end{tikzcd}$$

or, equivalently, in the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, tail, "u"'] & BA\prod_{B\Omega D} D \arrow[r] \arrow[d] \arrow[dr, phantom, "cart."] & BA \arrow[d, two heads, "Bp"] \\ F \arrow[r] \arrow[urr, dotted] & D \arrow[r] & B\Omega D. \end{tikzcd}$$

Such a lifting exists thanks to the adjunction between Ω and B and the lifting axiom (CM4) of the closed model category structure on Alg.

Fibrant and cofibrant objects

All the objects of Cogc are cofibrant since the cofibrations are the monomorphisms.

Let us show that an object of Cogc is fibrant if and only if it is isomorphic, as a graded coalgebra, to a reduced tensor coalgebra.

Let C be a fibrant object of Cogc. By the lifting axiom (CM4), the trivial cofibration $\psi : C \to B\Omega C$ admits a retraction r in Cogc. Denote by $p_1 : B\Omega C \to (B\Omega C)_{[1]}$ the canonical projection and set $p_1^C = r_{[1]} \circ p_1 \circ \psi$. It is easily checked that the morphism $p_1^C : C \to C_{[1]}$ is universal among the morphisms of graded objects $C' \to C_{[1]}$, where C' is a cocomplete graded coalgebra. Thus p_1^C induces an isomorphism of graded coalgebras

$$C \xrightarrow{\sim} \overline{T^c}(C_{[1]}).$$

The inverse uses the results of Section 1.3.3. We state the two results from this section that will be useful here.

(1.3.3.1) *The category* Alg_∞ *can be equipped with a model category structure in which the class of weak equivalences consists exactly of the* A_∞*-quasi-isomorphisms, and the class of cofibrations (respectively, fibrations) is formed by the morphisms* $f : A \to A'$*, where* A *and* A' *are* A_∞*-algebras, such that* f_1 *is a monomorphism (respectively, an epimorphism)* .

(1.3.3.5. *a*) *A morphism* f *is a weak equivalence in* Alg_∞ *if and only if its bar construction* Bf *is a weak equivalence in* Cogc.

Our proof of (1.3.3.1) is based on obstruction theory (see B.1). Therefore, we can interpret the reciprocal that we are going to prove as a consequence of the fact that the operad of A$_\infty$-algebras is the minimal cofibrant model, in the sense of M. Markl [Mar96], of the operad of associative algebras (see the introduction to Appendix B.1).

Let's assume that C is a coalgebra that is isomorphic, as a graded coalgebra, to a reduced tensor coalgebra. We want to show that it is fibrant. It's worth noting that the subcategory of Cogc formed by such coalgebras is equivalent to the category Alg_∞ of A$_\infty$-algebras. The coalgebra $B\Omega C$ also belongs to this subcategory. The morphism $C \to B\Omega C$ is a weak equivalence in Cogc. By Proposition (1.3.3.5. a), it induces a quasi-isomorphism in the primitives. Axiom (CM4) of Theorem (1.3.3.1) provides us with a lifting in the diagram

$$\begin{tikzcd} C \arrow[r, "\mathbf{1}"] \arrow[d, tail] & C \arrow[d] \\ B\Omega C \arrow[r] \arrow[ur, dotted] & 0. \end{tikzcd}$$

The coalgebra C is thus a retract of $B\Omega C$. Since the bar construction preserves fibrations, and as ΩC is a fibrant algebra, the coalgebra $B\Omega C$ is a fibrant coalgebra. A retract of a fibrant coalgebra is also fibrant, so the coalgebra C is fibrant.

Proof of point b *of Theorem 1.3.1.2.* This is a corollary of Lemma 1.3.2.3 which tells us that the adjunction morphisms $C \to B\Omega C$, where C is a coalgebra, and $\Omega BA \to A$, where A is an algebra, are weak equivalences in Cogc and in Alg. $\square$

The following lemma completes the proof above.

Lemma 1.3.2.7. Let A be an algebra and D a coalgebra. Consider a fibration $p : A \twoheadrightarrow \Omega D$ of Alg. The morphism $j : BA \prod_{B\Omega D} D \to BA$ of coalgebras of the Cartesian diagram

$$\begin{tikzcd} BA \prod_{B\Omega D} D \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "cart."] & D \arrow[d] \\ BA \arrow[r, "Bp"'] & B\Omega D. \end{tikzcd}$$

is a trivial cofibration of Cogc.

Proof. We will provide filtration on the coalgebras

$$BA \prod_{B\Omega D} D \qquad \text{and} \qquad BA$$

such that they are admissible filtered coalgebras, and such that j is a filtered quasi-isomorphism.

Consider the exact sequence of complexes:

$$0 \to K \to A \overset{p}{\twoheadrightarrow} \Omega D \to 0.$$

Since the algebra ΩD is free, we have a splitting of p in the category of graded algebras. The differential of

$$A \xrightarrow{\sim} K \oplus \Omega D$$

is then given by a matrix
$$\begin{bmatrix} d_K & d' \\ 0 & d_{\Omega D} \end{bmatrix}.$$
The splitting provides us with isomorphisms of graded coalgebras:
$$BA \xrightarrow{\sim} BK \prod B\Omega D,$$
$$BA \prod_{B\Omega D} D \xrightarrow{\sim} BK \prod D.$$
Equip the coalgebra $B\Omega D$ with the D-primitive filtration. We define filtrations on BA and $BA \prod_{B\Omega D} D$ by the sequences
$$(BA)_j = \sum_{p+q=j} (BK)_{[p]} \prod (B\Omega D)_q, \quad j \in \mathbf{N},$$
$$(BA \prod_{B\Omega D} D)_j = \sum_{p+q=j} (BK)_{[p]} \prod D_{[q]}, \quad j \in \mathbf{N}.$$
They are admissible and respect the differentials of the coalgebras BA and $BA \prod_{B\Omega D} D$. For these filtrations, the morphisms j is a filtered morphism. Let $j \geq 1$. As a graded object, the complex $\mathsf{Gr}(BA)$ is the sum of
$$\text{(I)} \qquad \mathsf{Gr}(B\Omega D) \otimes K^{\otimes p_1} \otimes \ldots \otimes \mathsf{Gr}(B\Omega D) \otimes K^{\otimes p_k} \quad , \quad k \geq 1.$$
The differential of $\mathsf{Gr}(BA)$ is constructed from the differentials of K, $\mathsf{Gr}D$ and the morphism $d' : \Omega D \to K$. As a graded object, the complex
$$\mathsf{Gr}(BA \prod_{B\Omega D} D) \xrightarrow{\sim} \mathsf{Gr}(BA) \prod \mathsf{Gr}D$$
is the sum of
$$\text{(II)} \qquad \mathsf{Gr}D \otimes K^{\otimes p_1} \otimes \ldots \otimes \mathsf{Gr}D \otimes K^{\otimes p_k} \quad , \quad k \geq 1.$$
The differential of $\mathsf{Gr}(BA \prod_{B\Omega D} D)$ is constructed from the differentials of K, $\mathsf{Gr}(B\Omega D)$ and the morphism $d' : \Omega D \to K$. Thus, the "naive" differential on the sum of the terms (I), respectively (II), is perturbed by the contribution of $d' : \Omega D \to K$. To show that j nevertheless induces a quasi-isomorphism between the sums, we introduce an additional filtration such that in the associated graded objects, the contribution of $d' : \Omega D \to K$ vanishes. Let the filtration $F_l\mathsf{Gr}(BA)$, $l \in \mathbf{N}$, of $\mathsf{Gr}(BA)$ be induced by
$$(BA)_l = BK \prod (B\Omega D)_{[l]}, \quad l \in \mathbf{N}.$$
Let the filtration $F_l\mathsf{Gr}(BA \prod_{B\Omega D} D)$, $l \in \mathbf{N}$, of $\mathsf{Gr}(BA \prod_{B\Omega D} D)$ whose l-th sub-object, $l \in \mathbf{N}$, is the sum of objects of type (II) comprising a number of terms $\mathsf{Gr}D$ less than or equal to l. The morphism
$$\mathsf{Gr}j : \mathsf{Gr}(BA \prod_{B\Omega D} D) \to \mathsf{Gr}(BA)$$
induces the morphisms
$$F_l\mathsf{Gr}(BA \prod_{B\Omega D} D) \to F_l\mathsf{Gr}(BA), \quad l \in \mathbf{N}.$$

It therefore induces a morphism between the graded objects associated to the filtrations according to the index l. The latter has as components the morphisms of complexes (with "naive" differentials)

$$\begin{array}{c}\mathsf{Gr}_{q_1} D \otimes K^{\otimes p_1} \otimes \ldots \otimes \mathsf{Gr}_{q_{k-1}} D \otimes K^{\otimes p_k} \\ \downarrow \\ \mathsf{Gr}_{q_1}(B\Omega D) \otimes K^{\otimes p_1} \otimes \ldots \otimes \mathsf{Gr}_{q_{k-1}}(B\Omega D) \otimes K^{\otimes p_k}\end{array}$$

which are quasi-isomorphisms (see Lemma 1.3.2.3). The morphism

$$\mathsf{Gr} j : \mathsf{Gr}(BA \prod_{B\Omega D} D) \xrightarrow{\sim} \mathsf{Gr}(BA)$$

is therefore a quasi-isomorphism. We have thus just shown that j is a filtered quasi-isomorphism of admissible coalgebras. By Lemma 1.3.2.2, the morphism j is a weak equivalence. It is a cofibration because it is clearly a monomorphism. $\square$

1.3.3 Alg_∞ as a "model category without limits"

In the category Alg_∞ of A_∞-algebras, we consider the following three classes of morphisms:

- the class $\mathcal{E}q$ is made up of *weak equivalences*, i.e. the morphisms $f : A \to A'$ such that f_1 is a quasi-isomorphism,
- the class $\mathcal{C}of$ is made up of the *cofibrations*, i.e. the morphisms $f : A \to A'$ such that f_1 is a monomorphism,
- the class $\mathcal{F}ib$ is made up of the *fibrations*, i.e. the morphisms $f : A \to A'$ such that f_1 is an epimorphism.

Theorem 1.3.3.1. The category Alg_∞, equipped with the three classes defined above, satisfies the axiom (A) below and the axioms (CM2) – (CM5) of Definition A.7. All objects are fibrant and cofibrant.

(A) Let $q : A \twoheadrightarrow A'$ be a fibration and $f : A'' \to A'$ a morphism. There exists a fiber product above

$$A \xrightarrow{q} A' \xleftarrow{f} A'' .$$

Axiom (A) is a weakening of axiom (CM1) of Definition A.7. Our proof of this theorem is entirely based on obstruction theory (Section B.1).

Lemma 1.3.3.2. Let A be an A_∞-algebra and K a complex considered as an A_∞-algebra (Remark 1.2.1.4). Suppose that the complex K is contractible. Let $g : (A, m_1^A) \to (K, m_1^K)$ be a morphism of complexes. There exists a morphism of A_∞-algebras

$$f : A \longrightarrow K$$

such that $f_1 = g$.

Proof. We construct by induction the morphisms

$$f_i : A^{\otimes i} \to K, \quad i \geq 1.$$

Let $f_1 = g$. Suppose that we have already constructed morphisms f_i, $1 \leq i \leq n$, which define an A_n-morphism $A \to K$. We are looking for a morphism f_{n+1} whose boundary is the cycle $-r(f_1, \ldots, f_n)$, i.e.

$$\delta(f_{n+1}) + r(f_1, \ldots, f_n) = 0 \quad \text{(see B.1.5)}.$$

As (K, m_1^K) is contractible, there exists such a morphism f_{n+1}. $\square$

Lemma 1.3.3.3.

a. Let $j : A \to D$ be a cofibration of Alg_∞. There exists an A_∞-algebra D' and an isomorphism of A_∞-algebras $k : D \to D'$ such that the composition $k \circ j : A \to D'$ is a strict morphism.

b. Let $q : C \to E$ be a fibration of Alg_∞. There exists an A_∞-algebra C' and an isomorphism $l : C' \to C$ such that the composition $q \circ l : C' \to E$ is a strict morphism.

Proof. a. We construct, by recursion, the morphisms

$$k_i : D^{\otimes i} \to D, \quad i \geq 1,$$

homogeneous of degree $1 - i$ such that $k \circ j$ is a strict morphism. We set $k_1 = \mathbf{1}_D$. Suppose we have already constructed morphisms k_i, $1 \leq i \leq n$, such that the equation

$$(eq_m) \qquad \sum_{1 \leq l \leq m} \; \sum_{\sum i_r = m} (-1)^s k_l \circ (j_{i_1} \otimes \ldots \otimes j_{i_l}) = 0, \quad 2 \leq m \leq n,$$

where s is the sign appearing in 1.2.1.2, is satisfied for all $2 \leq m \leq n$. Let r be a retraction in $\mathcal{G}r\mathsf{C}$ of $j_1 : A \to D$. Let k_{n+1} be the morphism defined by the sum

$$-\left[\sum_{1 \leq l \leq n} \; \sum_{\sum i_r = n+1} (-1)^s k_l \circ (j_{i_1} \otimes \ldots \otimes j_{i_l}) \right] \circ r^{\otimes n+1}.$$

The sequence $(k_1, \ldots, k_{n+1})$ satisfies the equation (eq_m) for $2 \leq m \leq n+1$. Since k_1 is an isomorphism of graded objects, the morphisms k_i, $i \geq 1$, induces an isomorphism

$$K : \overline{T^c}(SD) \xrightarrow{\sim} \overline{T^c}(SD).$$

We define D' as the A_∞-algebra with the underlying graded object D and with multiplications m'_i, $i \geq 1$, defined using the bijections $m'_i \leftrightarrow b'_i$ (see 1.2.2), by the equations

$$b'_i = (K \circ b \circ K^{-1})_i, \quad i \geq 1.$$

Then the morphism $k : D \to D'$ is clearly an isomorphism of Alg_∞ and the composition $k \circ j$ is strict by construction of k.

b. The proof is similar. A section of q_1 must be used instead of a retraction of j_1. $\square$

Proof of Theorem 1.3.3.1 :

(A) : Let $q : A \twoheadrightarrow A'$ be a fibration and $f : A'' \to A'$ a morphism of Alg_∞. The bar construction sends the morphisms q and f to the morphisms $Q : BA \to BA'$ and $F : BA'' \to BA'$. We will show that the fiber product of Cogc over

$$BA \overset{Q}{\twoheadrightarrow} BA' \xleftarrow{F} BA''$$

is still a reduced tensor coalgebra in $\mathcal{G}r\mathsf{C}$ A section of Q_1 in $\mathcal{G}r\mathsf{C}$ induces an isomorphism

$$SA \xrightarrow{\sim} SA' \oplus K,$$

where K is the kernel of Q_1. The fiber product $BA \prod_{BA'} BA''$ is isomorphic, as a graded coalgebra, to

$$\overline{T^c}K \prod \overline{T^c}(SA'') \xrightarrow{\sim} \overline{T^c}(K \oplus SA'').$$

(CM2) and (CM3) : Immediate.

(CM4) *Lifting* : Consider the diagram of A_∞-algebras

$$\text{(I)} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "j"'] & C \arrow[d, two heads, "q"] \\ D \arrow[r, "g"'] & E, \end{tikzcd}$$

where q is a fibration and j is a cofibration. By Lemma 1.3.3.3, by replacing this diagram with an isomorphic one, we can assume that the morphisms j and q are strict. Suppose that the fibration q (resp. the cofibration j) is trivial. We are looking for a lifting α that makes the two triangles of the diagram commute

$$\text{(I}^+\text{)} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "j"'] & C \arrow[d, two heads, "q"] \\ D \arrow[r, "g"'] \arrow[ur, "\alpha"] & E. \end{tikzcd}$$

We will construct, by recursion, the corresponding morphisms

$$\alpha_i : D^{\otimes i} \to C, \quad i \geq 1.$$

By point a of axiom (CM4) for the model category $\mathcal{C}\mathsf{C}$, there exists a lifting α_1 that makes both triangles commute

$$\text{(II)} \qquad \begin{tikzcd} (A, m_1^A) \arrow[r, "f_1"] \arrow[d, tail, "j_1"'] & (C, m_1^C) \arrow[d, two heads, "q_1"] \\ (D, m_1^D) \arrow[r, "g_1"'] \arrow[ur, dotted, "\alpha_1"] & (E, m_1^E). \end{tikzcd}$$

Suppose that we have already constructed morphisms α_i, $1 \le i \le n$, such that the diagram (I^+) commutes in the diagram of A_n-algebras. We need to find an α_{n+1} such that

$$\begin{array}{ll} (1) & \delta(\alpha_{n+1}) + r(\alpha_1, \ldots, \alpha_n) = 0, \quad \text{(see B.1.5)} \\ (2) & \alpha_{n+1} \cdot {j_1}^{\otimes n+1} = f_{n+1}, \\ (3) & q_1 \cdot \alpha_{n+1} = g_{n+1}. \end{array}$$

Choose a solution β to (2) and (3). For example, if ρ is a retraction of j_1 and σ a section of q_1 in $\mathcal{G}r\mathsf{C}$, we can choose

$$\beta = f_{n+1}\rho^{\otimes n+1} + \sigma g_{n+1} - \sigma q_1 f_{n+1}\rho^{\otimes n+1}.$$

The morphism j is strict By Lemma B.1.6, we have

$$(\delta(\beta) + r(\alpha_1, \ldots, \alpha_n)) \circ j_1 = \delta(\beta \circ j_1) + r(\alpha_1 \circ j_1, \ldots, \alpha_n \circ {j_1}^{\otimes n}),$$

and the term on the right is equal to

$$\delta(f_{n+1}) + r(f_1 \circ j_1, \ldots, f_n) = 0.$$

Similarly, we have $q_1 \circ (\delta(\beta) + r(\alpha_1, \ldots, \alpha_n)) = 0$. The cycle $\delta(\beta) + r(\alpha_1, \ldots, \alpha_n)$ is factored into

$$D^{\otimes n+1} \xrightarrow{p} \mathsf{cok}\, {j_1}^{\otimes n+1} \xrightarrow{c'} \mathsf{ker}\, q_1 \xrightarrow{i} C,$$

where p is the canonical projection and i the canonical injection. Since $\mathsf{ker}\, q_1$ (res. $\mathsf{cok}({j_1}^{\otimes n+1})$) is contractible, the cycle c' is the boundary of a morphism h'. The morphism $\alpha_{n+1} = \beta - i \circ h' \circ p$ satisfies the equations (1), (2) and (3).

Remark 1.3.3.4. The proof of the lifting axiom (CM4) shows that for any lifting α_1 in the category $\mathcal{C}\mathsf{C}$ of diagram (II), there exists a lifting $\alpha : D \to C$ in the diagram (I).

(CM5) *factorization:* Let $f : A \to B$ be a morphism of A_∞-algebras.

a. Let $C = B \oplus S^{-1}B$ be the cone of the identity of $S^{-1}B$. Consider the complex C as an A_∞-algebra (See 1.2.1.4). Let $j : A \to A \prod C$ be the morphism of A_∞-algebras with components $\mathbf{1}_A$ and 0. The morphism $q_1 : A \oplus C \to B$ with components the morphism f and the canonical projection $C \to B$ is a lifting in the diagram in $\mathcal{C}\mathsf{C}$.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, tail, "j_1"] & B \arrow[d, two heads] \\ A \oplus C \arrow[r] \arrow[ur, dotted] & 0. \end{tikzcd}$$

Remark 1.3.3.4, applied to point a of axiom (CM4), provides us with a lifting in the diagram in Alg_∞.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "j"] & B \arrow[d, two heads] \\ A \prod C \arrow[r] \arrow[ur, dotted, "q"] & 0. \end{tikzcd}$$

In the factorization $f = q \circ j$, j is a trivial cofibration, and q is a fibration.

b. Let $C = SA \oplus A$ be the cone of the identity on the complex (A, m_1). Let's consider C as an A_∞-algebra. By Lemma 1.3.3.2, there exists a morphism of A_∞-algebras $i : A \to C$ such that i_1 is the canonical injection $A \to C$. Let $j : A \to B \prod C$ be the morphism of A_∞-algebras with components f and i. It is a trivial cofibration. Let q be the canonical projection $B \prod C \to B$. It is a fibration, and the morphism f factors as $q \circ j$.

□

Links between the "model categories without limits" Alg_∞ and the model category Cogc

Let Cogtr be the subcategory of Cogc consisting of coalgebras that are reduced tensor coalgebras as graded coalgebras. The bar construction induces an isomorphism of categories $\mathsf{Alg}_\infty \to \mathsf{Cogtr}$. Equip Cogtr with the structure of a "limitless model category" given by this isomorphism. Therefore, weak equivalences (resp. cofibrations, resp. fibrations) are the morphisms $F : (\overline{T^c}V, b) \to (\overline{T^c}V', b')$ that induce in the primitives a quasi-isomorphism $F_1 : (V, b_1) \to (V', b'_1)$ (resp. a monomorphism, resp. an epimorphism).

Proposition 1.3.3.5. Let A and A' be two A_∞-algebras.

a. A morphism $f : BA \to BA'$ is a weak equivalence of Cogtr if and only if it is a weak equivalence in Cogc.

b. A morphism $j : BA \to BA'$ is a cofibration of Cogtr if and only if it is a cofibration in Cogc.

c. A morphism $q : BA \to BA'$ is a fibration of Cogtr if and only if it is a fibration in Cogc.

Let's begin with a lemma.

Lemma 1.3.3.6. Let A be an A_∞-algebra. The morphism $\phi : BA \to B\Omega BA$ is a weak equivalence in Cogtr.

Proof. We want to show that the morphism $\phi_{[1]}$ is a quasi-isomorphism, or equivalently, that the morphism

$$S^{-1}\phi_{[1]} : (A, m_1) \to \Omega BA$$

is a quasi-isomorphism. The morphism $S^{-1}\phi_{[1]}$ is the canonical injection of A into ΩBA. Equip ΩBA with the filtration induced by the primitive filtration of BA. Just as at the end of the proof of point b of Lemma 1.3.2.3, we show that

$$\mathsf{Gr}_0(\Omega BA) = A \quad \text{and} \quad \mathsf{Gr}_i(\Omega BA) = 0 \quad \text{for} \quad i \geq 1.$$

□

Proof of Proposition 1.3.3.5. a. Let $f : BA \to BA'$ be a weak equivalence in Cogtr. The morphism f is clearly a filtered quasi-isomorphism for the primitive filtration. Therefore, it is a weak equivalence in Cogc. Let's assume that f is a weak equivalence in Cogc. By the definition of weak equivalences in Cogc, the morphism Ωf is a quasi-isomorphism, and consequently, the morphism $B\Omega f$ is a weak equivalence in Cogtr. By Lemma 1.3.3.6, the two horizontal arrows in the commutative diagram

$$\begin{tikzcd} BA \arrow[r] \arrow[d, "f"'] & B\Omega BA \arrow[d, "B\Omega f"] \\ BA' \arrow[r] & B\Omega B'A', \end{tikzcd}$$

are weak equivalences in Cogtr, and thus, f is also a weak equivalence in Cogtr.

b. Since the cofibrations in Cogc are monomorphisms, a cofibration in Cogtr is also a cofibration in Cogc. Conversely, if $j : BA \to BA'$ is a cofibration in Cogc, its restriction to the primitives $(BA)_{[1]} = SA$ is a monomorphism. Since we have $f((BA)_{[1]}) \subset (BA')_{[1]}$, the morphism $j_{[1]} : SA \to SA'$ is a monomorphism, and therefore, j is a cofibration in Cogtr.

c. We recall that the fibrations in a model category are the morphisms that have the right lifting property with respect to trivial cofibrations. This fact follows from axioms (CM5) and (CM3) and holds true for Cogtr as well. By points a and b, a fibration in Cogc is also a fibration in Cogtr. Let's assume that q is a fibration in Cogtr. Consider the diagram in Cogc

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, tail, "j"'] & BA \arrow[d, "q"] \\ C' \arrow[r, "g"'] & BA', \end{tikzcd}$$

where j is a trivial cofibration in Cogc. We are looking for a lift of g relative to f. In the diagram in Cogc below

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dd, tail, "j"'] \arrow[dr, tail, "\phi"] & & BA \arrow[dd, "q"] \\ & B\Omega C \arrow[dd, tail] & \\ C' \arrow[rr, "g"] \arrow[dr, tail] & & BA', \\ & B\Omega C' & \end{tikzcd}$$

ϕ is a trivial cofibration in Cogc and BA is fibrant in Cogc. Therefore, we have a factorization of f as $f' \circ \phi$ for a morphism $f' : B\Omega C \to BA$. Since Ωj is a monomorphism and a quasi-isomorphism, the morphism $B\Omega j$ is a trivial cofibration in Cogtr. As BA' is cofibrant in Cogtr, the morphism $q \circ f'$ factors as $g' \circ B\Omega f$ for a morphism $g' : B\Omega C' \to BA'$. Hence, it suffices to find a lift of g' relative to f'. According to the axiom (CM4) for the category Cogtr, there exists one. $\square$

1.3.4 Homotopy in the classical sense

Let C and C' be two cocomplete coalgebras. Let f and g be two coalgebra morphisms $C \to C'$. They are *homotopic in the classical sense* if there exists a (f, g)-coderivation $h : C \to C'$ of degree -1 such that $\delta(h) = f - g$. We compare this notion to the notion of homotopy in the sense of model categories (see appendix A).

Proposition 1.3.4.1. Let c and c' be two cocomplete coalgebras and f, g be two morphisms $c \to c'$.

a. if f and g are homotopic in the classical sense, they are left homotopic (see the definition A.9).

b. if the coalgebra c' is fibrant, then f and g are homotopic in the classical sense if and only if they are left homotopic.

Proof. *a.* We will construct a cylinder $C \wedge I$ for the coalgebra C, and then we will show that the classical notion of homotopy is equivalent to the notion of $C \wedge I$-homotopy on the left.

We denote I as the complex with the degree 0 component as $e \oplus e$, the degree -1 component as e, and all other components are zero. We denote e_0 and e_1 as the components of I_0. The differential $d : I \to I$ is given by:

$$d_1 = \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} : e \to e_0 \oplus e_1.$$

Let $\Delta : I \to I \otimes I$ be the morphism whose non-zero components are given by the morphisms

$$e_0 \xrightarrow{\sim} e_0 \otimes e_0, \quad e_1 \xrightarrow{\sim} e_1 \otimes e_1, \quad e \xrightarrow{\sim} e_0 \otimes e, \quad e \xrightarrow{\sim} e \otimes e_1$$

given by the unital constraint of the base monoidal category (1.1.1). This defines a coassociative, differential graded coalgebra structure on I.

Let C be a cocomplete coalgebra. The tensor product $C \otimes I$ naturally inherits a differential graded coalgebra structure the by comultiplication $C \otimes I \to (C \otimes I) \otimes C \otimes I \simeq C \otimes C \otimes I \otimes I$. It is cocomplete. We denote C_0 and C_1 as the components of $C \coprod C$. We define the cylinder $C \wedge I = C \otimes I$ for C using the two morphisms of differential graded coalgebras i and p

$$C_0 \coprod C_1 \xrightarrow{i} C \otimes I \xrightarrow{p} C,$$

where the morphism i has nonzero components

$$C_0 \xrightarrow{\sim} C \otimes e_0, \quad C_1 \xrightarrow{\sim} C \otimes e_1,$$

and where the morphism p has nonzero components

$$C \otimes e_0 \xrightarrow{\sim} C, \quad C \otimes e_1 \xrightarrow{\sim} C,$$

given by the unital constraints of the base category. The morphism i is a cofibration and the morphism p is a weak equivalence.

Let C' be a cocomplete coalgebra. Let f, g and h be three graded morphisms $C \to C'$, with degrees 0, 0, and -1, respectively.

Consider the graded morphism of degree 0, $H : C \otimes I \to B$, whose components are the three graded morphisms

$$C \otimes e_0 \simeq C \xrightarrow{f} C', \quad C \otimes e_1 \simeq C \xrightarrow{g} C'$$

$$\text{and} \quad C \otimes e \simeq C \xrightarrow{h} C'.$$

The morphism $H : C \otimes I \to C'$ is a morphism of coalgebras if and only if

- the morphisms f and g are morphisms of coalgebras $C \to C'$,
- the morphism $h : C \to C'$ is a (f, g)-coderivation.

It is compatible with differentials if and only if

- the morphsism f and g are morphisms of complexes $C \to C'$
- the morphism $h : C \to C'$ realizes a homotopy between the morphisms of complexes f and g

Finally, we verify that the morphism H is indeed a $C \wedge I$-homotopy between f and g.

b. Let C' be a fibrant coalgebra. Let f and $g : C \to C'$ be two homotopy equivalent morphisms in the category of models. Let $C \wedge I$ still denote the cylinder constructed above. By Lemma A.12, there exists a left $C \wedge I$-homotopy $H : C \wedge I \to C'$ between f and g. By the proof of point a, there exists a homotopy $h : C \to C'$ in the classical sense between f and g. □

1.3.5 Weak equivalences and quasi-isomorphisms

We denote by Qis the class of quasi-isomorphisms of Cogc and by $Qisf$ the class of morphisms $f : C \to D$ of Cogc such that C and D admit admissible filtrations for which f is a filtered quasi-isomorphism.

This section is devoted to the comparison of the three classes $\mathcal{E}q$, Qis and $Qisf$. We will show in particular the following inclusions

$$Qisf \subseteq \mathcal{E}q \subset Qis.$$

Proposition 1.3.5.1.

a. We have the inclusion $Qisf \subseteq \mathcal{E}q$. On the other hand, the canonical functor

$$\mathsf{Cogc}[Qisf^{-1}] \longrightarrow \mathsf{Cogc}[\mathcal{E}q^{-1}] = \mathsf{Ho}\,\mathsf{Cogc}$$

is an equivalence.

b. The weak equivalences of Cogc are quasi-isomorphisms.

c. The class $\mathcal{E}q$ is strictly included in the class Qis.

d. Let C and D be two objects in Cogc concentrated in degrees < -1. Any quasi-isomorphism of coalgebras $C \to D$ si a weak equivalence.

e. Let C and D be two objects of Cogc concentrated in degrees ≥ 0. Every quasi-isomorphism of coalgebras $C \to D$ is a weak equivalence.

Proof. a. Recall (1.3.2.2) that a filtered quasi-isomorphism of coalgebras is a weak equivalence in Cogc. We thus need to show that weak equivalences become isomorphisms in the localized category $\mathsf{Cogc}[Qisf^{-1}]$. Let $f : C \to C'$ be a weak equivalence in Cogc. The morphism

$$\Omega f : \Omega C \to \Omega C'$$

is therefore a quasi-isomorphism of algebras. By Lemma 1.3.2.3, the morphism $B\Omega f : B\Omega C \to B\Omega C'$ is a filtered quasi-isomorphism. Recall from Lemma 1.3.2.3 that the adjunction morphisms $C \to B\Omega C$ and $D \to B\Omega D$ are filtered quasi-isomorphisms. We deduce the commutative diagram in Cogc

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] & B\Omega C \arrow[d, "B\Omega f"] \\ C' \arrow[r] & B\Omega C' \end{tikzcd}$$

that the morphism f becomes an isomorphism in the category $\mathsf{Cogc}[Qisf^{-1}]$.

b. Filtered quasi-isomorphisms are quasi-isomorphisms. The saturation property of the class Qis, applied to the diagram above, shows that a weak equivalence is a quasi-isomorphism.

c. We will construct an example of a coalgebra that is acyclic but not weakly equivalent to the zero coalgebra.

Let A be a nonzero unital algebra in the base category C. Consider A as an associative algebra (forgetting the unit), that is, as an object in the category Alg from Theorem (1.3.1.2).

Since A is not quasi-isomorphic to the zero algebra, the coalgebra $BA = (\overline{T^c}SA, b)$ is not weakly equivalent to the zero coalgebra (1.3.1.2, *b*). However, it is ineeed quasi-isomrphic to the zero coalgebra: in fact, the complex underlying $S^{-1}BA$ is the complex

$$\cdots \to A \otimes A \otimes A \to A \otimes A \to A \to 0,$$

which is isomorphic to the bar resolution of the algebra A. This complex is acyclic because A is unital (see [CE99, IX.6] where this complex is called the "standard resolution").

d. Let C and D be two cocomplete coalgebras concentrated in degrees < -1. We will show that the morphism $\Omega f : \Omega C \to \Omega D$ is a quasi-isomorphism in Alg. Endow ΩC (resp. ΩD) with the decreasing filtration given by

$$F_l \Omega C = \bigoplus_{p \geq l} (S^{-1}C)^{\otimes p} \quad \left(\text{resp. } F_l \Omega D = \bigoplus_{p \geq l} (S^{-1}D)^{\otimes p} \right), \quad l \in \mathbf{N}.$$

By our assumption, the morphism Ωf induces quasi-isomorphisms in the subquotients of these filtrations. It follows that, for all $n \in \mathbf{N}$, it induces an isomorphism in H^{-n}, since we have

$$(F_l \Omega C)^n = (F_l \Omega D)^n = 0 \quad \text{for} \quad l > n,$$

according to the asssumption about C and D.

e. The proof is the same as for point *d.* It is enough to note that the complex $S^{-1}C$ is concentrated in degrees > 0 (instead of < 0). □

1.4 Transfer of structures along homotopy equivalences

The goal of this section is to (re)show the *minimal model theorem* (Corollary 1.4.1.4).

1.4.1 Minimal model

Theorem 1.4.1.1. Let A be an A_∞-algebra. Consider a homotopy equivalence in $\mathcal{C}\mathsf{C}$

$$g : (V, d) \to (A, m_1^A),$$

where (V, d) is a complex. There exists an A_∞-algebra structure on V such that $m_1^V = d$ and a morphism of A_∞-algebras

$$f : V \to A$$

such that $f_1 = g$.

This result has been known since the 1970s in the case of a connected A_∞-algebra (i.e., concentrated in homological degrees ≥ 1) and a complex (V, d) where the differential d is zero (V is isomorphic to H^*A). There are two methods for proving this theorem, one using the "obstruction method" [Che77a], [Che77b], [Kad80], [Smi80], [Gug82] and one using the "tensor trick" [Hue86], [GS86], [GL89], [GLS91], [HK91], [Mer99], [KS01]. The article [JL01] presents the unification of these different methods. Here, we provide a proof using obstructions.

Proof. By axiom (CM5) for the model category $\mathcal{C}\mathsf{C}$, the morphism g factors as $q \circ j$, where q is a trivial fibration and where j is a trivial cofibration. It suffices to show the theorem in the case where the homotopy equivalence is an epimorphism and in the case where it is a monomorphism.

Suppose that g is a trivial fibration in $\mathcal{C}\mathsf{C}$. Let K be the kernel of g. Since K is contractible, we can split g in a category of complexes. This splitting induces an isomorphism of complexes

$$V \xrightarrow{\sim} K \oplus A$$

by which the morphism g is identified with the projection $K \oplus A \to A$. Consider K as an A_∞-algebra (see 1.2.1.4). Endow the underlying graded object of V with the A_∞-algebra structure of $K \prod A$. The morphism f is the canonical morphism $K \prod A \to A$ in Alg_∞.

Now suppose that g is a trivial cofibration in $\mathcal{C}\mathsf{C}$. Let K be the cokernel of g. Since it is contractible, we can split g in a category of complexes. This splitting induces an isomorphism in $\mathcal{C}\mathsf{C}$

$$A \xrightarrow{\sim} K \oplus V$$

by which the morphism g is identified with the injection $V \to K \oplus V$. Consider K as an A_∞-algebra. By Lemma 1.3.3.2, there exists a morphism of A_∞-algebras $h : A \to K$ such that h_1 is the projection $K \oplus V \to K$ in $\mathcal{C}\mathsf{C}$. Thanks to axiom (A) of Theorem 1.3.3.1, the morphism h admits a kernel in the category Alg_∞. The underlying graded object of $\ker h$ is V. Thus we have downed V with an A_∞-algebra structure such that m_1^V is the differential of V. The canonical morphism $V \to A$ is such that $f_1 = g$. □

Minimal model

Definition 1.4.1.2. An A_∞-algebra is *minimal* if $m_1 = 0$. Let A be an A_∞-algebra. A *minimal model for* A is a A_∞-quasi-isomorphism of A_∞-algebras $A' \to A$ where A' is minimal.

Remark 1.4.1.3. This use of the term "minimal model", due to M. Kontsevich, is different from the conventional usage in rational homotopy (Sullivan's minimal model). It can be justified by the fact that the bar construction BA' is a minimal model in the sense of H. J. Baues and J.-M. Lemaire [BL77] of the coalgebra BA. Note that a minimal model of BA does not in general give a minimal model of A: let $(\overline{T^c}SV, b)$ be a reduced tensor coalgebra on SV whose differential b induces zero in 1-primitives; if $(\overline{T^c}SV, b)$ is a minimal model of BA, i.e. if we have a *quasi-isomorphism* of coalgebras

$$F : (\overline{T^c}SV, b) \to BA,$$

the A_∞-algebra V such that $BV = (\overline{T^c}SV, b)$ is not in general a minimal model for the A_∞-algebra A. However, if F is a *weak equivalence* of Cogc, V is a minimal model of the A_∞-algebra A.

Corollary 1.4.1.4. Let A be an A_∞-algebra. There exists an A_∞-algebra structure on its homology H^*A such that

a. $m_1 = 0$ and m_2 are induced by m_2^A,

b. there exists a morphism of A_∞-algebras $H^*A \to A$ lifting the identity of H^*A.

This structure is unique up to (non-unique) isomorphism.

Proof. Since the base category C is semi-simple, we have an isomorphism in the category of complexes

$$(A, m_1^A) \xrightarrow{\sim} H^*A \oplus K$$

for a contractible complex K. The result is deduced from the theorem 1.4.1.1 applied to the canonical injection

$$g : H^*A \longrightarrow A.$$

The uniqueness of the structure comes from the fact that a morphism f between minimal A_∞-algebras is a quasi-isomorphism if and only if f_1 is an isomorphism if and only if f is an isomorphism. $\square$

1.4.2 Link with the perturbation lemma

A *perturbation* δ of the differential d of a filtered complex W is a graded morphism $\delta : W \to W$ of degree $+1$ that decrease the filtration and such that $d + \delta$ is still a differential, that is, such that

$$d \circ \delta + \delta \circ d + \delta^2 = 0.$$

A *contraction* [EML53] (see also [HK91] and the referenecs given in [HK91])

$$\Big(V \underset{i}{\overset{\rho}{\leftrightarrows}} W \ , H \Big)$$

is given by two complexes V and W, two morphisms de complexes $i : V \to W$ and $\rho : V \to W$ and a graded morphism $H : W \to W$ of degree -1 such that

$$\rho \circ i = \mathbf{1}_V, \quad i \circ \rho = \mathbf{1}_W + \delta(H), \quad H \circ i = 0, \quad \rho \circ H = 0 \quad \text{nad} \quad H^2 = 0.$$

We also say that W *contracts onto* V If the complexes are filtered, the contraction is *filtered* if the morphisms are filtered relative to these filtrations.

Let V and W be complexes equipped with exhaustive filtrations and let

$$\Big((V, d_V) \underset{i}{\overset{\rho}{\leftrightarrows}} (W, d_W) \ , H \Big)$$

be a filtered contraction and δ a perturbation of the differential d_W. The perturbation lemma ([Gug72], [HK91]) gives a new differential d_V^δ of V and morphisms i^δ, ρ^δ and H^δ such that

$$\Big((V, d_V^\delta) \underset{i^\delta}{\overset{\rho^\delta}{\leftrightarrows}} (W, d_W + \delta) \ , H^\delta \Big)$$

is a filterd contraction. Suppose that the filtered contraction above is a *filtered contraction of coalgebras* : the objects V and W are filgered differential graded coalgebras, the morphisms i and ρ are filtered coalgebra morphisms, H is a $\mathbf{1}$-$(i\rho)$-filtered coderivation of W. Suppose also that the perturbation δ is a perturbation of differential coalgebras, i.e. δ is a $\mathbf{1}$-$\mathbf{1}$-coderivation of W. The perturbation lemma then produces a contraction of coalgebras ([HK91], [GS86], [GL89], [GLS91], [Mer99]).

Let A be an A_∞-algebra and let

$$0 \longrightarrow (V, d_V) \underset{i}{\overset{\rho}{\leftrightarrows}} (A, m_1) \underset{p}{\overset{\sigma}{\leftrightarrows}} (K, d_K) \longrightarrow 0$$

be a split exact sequence of complexes such that

$$\rho \circ \sigma = 0 \quad \text{and} \quad i \circ \rho + \sigma \circ p = \mathbf{1}_A.$$

Let h be a contracting homotopy of K such that $h^2 = 0$. From this data, we have two natural ways to define an A_∞-algebra structure on V and an A_∞-morphism

$$V \to A$$

whose first component is i.

First method : the perturbation lemma
We apply the perturbation to the filtered contraction and to the perturbation of coalgebras

$$\left(\overline{T^c}S(V, d_V) \underset{F}{\overset{R}{\leftrightarrows}} \overline{T^c}S(A, m_1)\ , H\right) \quad \text{and} \quad \delta : \overline{T^c}SA \to \overline{T^c}SA,$$

where $F = \overline{T^c}Si$, $R = \overline{T^c}S\rho$, H is the unique $\mathbf{1}$-(FR)-coderivation lifting $\sigma \circ h \circ p$ and $\delta = b - b_1$ (here b is the differential of BA).

We obtain a new differential b' on $\overline{T^c}SV$ and a morphism of coalgebras

$$F^\delta : (\overline{T^c}SV, b') \to (\overline{T^c}SA, b).$$

We obtain an A_∞-algebra structure on V (denote this A_∞-algebra V^δ) and an A_∞-morphism

$$f^\delta : V^\delta \to A.$$

Second method: the kernel of the A_∞*-morphism* g
Define by recurrence the morphisms

$$g_i : A^{\otimes i} \to K, \quad i \geq 1,$$

by setting

$$g_1 = p \quad \text{and} \quad g_i = -h \circ r(g_1, \ldots, g_{i-1}), \quad i \geq 2,$$

where $r(g_1, \ldots, g_{i-1})$ is the cycle of lemma (B.1.5). The lemma (B.1.5) shows that they define an A_∞-morphism $g : A \to K$ (where K is the complex K considered as an A_∞-algebra). Axiom (A) of theorem (1.3.3.1) shows that there exists a kernel for g in the category Alg_∞

$$V^g = \mathsf{ker}\, g \to A.$$

Since the underlying graded object of the A$_\infty$-algebra V^g is V, this defines an A$_\infty$-structure on V and an A$_\infty$-morphism
$$f^g : V^g \to A.$$

Lemma 1.4.2.1. We have an isomorphism $\theta : V^\delta \to V^g$ such that $\theta_1 = \mathbf{1}$ and $f^\delta = f^g \circ \theta$.

Proof. Recall the descriptions of the A$_\infty$-structure of V^δ and of f^δ in terms of trees due to M. Kontsevich and Y. Soibelman [KS01, 6.4].

The A$_\infty$-structure of V^δ is defined by the following formulas:
$$m_1^\delta = 0, \quad m_2^\delta = \rho \circ m_2 \circ (i \otimes i), \quad m_i^\delta = \sum_{T \in \mathcal{T}} (-1)^s m_{i,T}, \quad i \geq 3,$$
where s and T, $\mathcal{T}$ and $m_{i,T}$ are defined by: Consider the set $\mathcal{T}$ of oriented planar trees T with $i+1$ terminal vertices (the root an the leaves), such that the arity $|v|$ of every internal vertex $v \in T$ (i.e. the number of arrows arriving at v) is ≥ 2. To describe the morphism
$$m_{i,T} : (V^\delta)^{\otimes i} \to V^\delta, \quad i \geq 3, \quad T \in \mathcal{T},$$
we need to consider the tree $\overline{T}$ constructed from T by adding an internal vertex in the middle of each internal edge. The tree $\overline{T}$ is thus composed of two types of internal vertices: the *old* ones corresponding to the internal vertices of T and the *new* ones that have just been added. We color the vertices of $\overline{T}$ with the following morphisms:

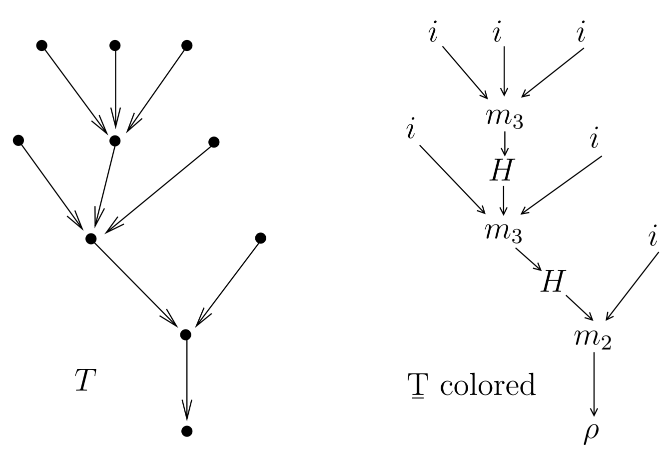
- ρ at the root,

- i at the leaves

- $m_{|v|}$ at the old internal vertices v (whose arity is $|v|$),

- H at the new internal vertices.

To each colored tree $\overline{T}$, we associated the morphism $m_{i,T}$, which consists of composing the colorings by descending along the tree from the leaves to the root. Here is an example:



The morphism $m_{6,T}$ is given by

$$\rho \circ m_2 \circ (H \otimes \mathbf{1}) \circ (m_3 \otimes \mathbf{1}) \circ (\mathbf{1} \otimes H \otimes \mathbf{1}^{\otimes 2}) \circ (\mathbf{1} \otimes m_3 \otimes \mathbf{1}^{\otimes 2}) \circ (i^{\otimes 6}).$$

The sign $(-1)^s$ associated to T is given by the equation

$$\begin{gathered}\rho \circ m_2 \circ (H \otimes \mathbf{1}) \circ (m_3 \otimes \mathbf{1}) \circ (\mathbf{1} \otimes H \otimes \mathbf{1}^{\otimes 2}) \circ (\mathbf{1} \otimes m_3 \otimes \mathbf{1}^{\otimes 2}) \circ (i^{\otimes 6}) \circ (\omega^{\otimes 6}) = \\ (-1)^s \omega \circ \rho' \circ b_2 \circ (H' \otimes \mathbf{1}) \circ (b_3 \otimes \mathbf{1}) \circ (\mathbf{1} \otimes H' \otimes \mathbf{1}^{\otimes 2}) \circ (\mathbf{1} \otimes b_3 \otimes \mathbf{1}^{\otimes 2}) \circ (i^{\otimes 6}),\end{gathered}$$

where

$$\rho' = s \circ \rho \circ \omega, \quad H' = -s \circ H \circ \omega \quad \text{and} \quad i' = s \circ i \circ \omega.$$

The sign in the general case is obtained in the same way.

The morphism $f^\delta : V^\delta \to A$ is given by the formulas

$$f_1^\delta = i, \quad f_i^\delta = \sum_{T \in \mathcal{T}} (-1)^s f_{i,T}, \quad i \geq 2,$$

where the morphsisms $f_{i,T}$ and the sign s are constructed in the same way by coloring the root of the tree $\overline{T}$ with H instead of ρ. Remark (1.4.2.2) below will show that the morphisms m_i^δ, $i \geq 1$, and f_i^δ, $i \geq 1$, indeed define A_∞-structures.

Note that the signs above are such that

$$b_i^\delta = \sum_{T \in \mathcal{T}} b_{i,T} \quad \text{and} \quad F_i^\delta = \sum_{T \in \mathcal{T}} F_{i,T}, \quad i \geq 1,$$

where $b_{i,T}$ and $F_{i,T}$ are obtained by coloring the vertices of the trees $\overline{T}$ with b_i (resp. i', ρ', H') on the vertices that were previously colored m_i (resp. i, ρ, H).

We will now specify that A_∞-morphism

$$g : A \to K$$

in terms of trees. A straightforward calculation (using the fact that $h^2 = 0$) shows that the morphism g_i, $i \geq 1$, is given by the formulas

$$g_1 = p \quad \text{and} \quad g_i = -p \circ h \circ m_i, \quad i \geq 2.$$

Since $h \circ p = p \circ H$, the morphisms g_i correspond to the colored trees (they do not necessarily belong to $\mathcal{T}$)

$$\begin{tikzcd} \mathbf{1} \arrow[d] \\ p \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mathbf{1} \arrow[drr] & \mathbf{1} \arrow[dr] & & \mathbf{1} \arrow[dl] & \mathbf{1} \arrow[dll] \\ & & m_i \arrow[d] & & \\ & & H \arrow[d] & & \\ & & p & & \end{tikzcd}$$

The sign appearing in the formula for g implies the equations

$$G_1 = p' \quad \text{and} \quad G_i = -p' \circ H' \circ b_i, \quad i \geq 2,$$

where $p' = s \circ p \circ \omega$.

We show that the composiiton $g \circ f^\delta$ is zero. It suffices to show the equalities

$$\sum_{\sum \alpha_k = n} G_i(F^\delta_{\alpha_1} \otimes \ldots \otimes F^\delta_{\alpha_i}) = 0, \quad n \geq 1.$$

Let $n \geq 1$. Since the G_i and the $F^\delta_{\alpha_k}$ are sums of compositions associated with colored trees, the above sum is the sum of compositions associated with concatenated colored trees. We check that the concatenated colored trees involved in the sums

$$\sum_{\sum \alpha_k = n,\ i \geq 2} G_i(F^\delta_{\alpha_1} \otimes \ldots \otimes F^\delta_{\alpha_i}) \quad \text{and} \quad G_1 \circ F^\delta_n$$

are the same. In the first sum, the sign in front of each composition associated with a concatenated colored tree is negative because, for $i \geq 2$, we have $G_i = -p' \circ H' \circ b_i$. In the second sum, it is positive because $G_1 = p'$. Thus, we have $G \circ F^\delta = 0$. The morphism f^δ factors as $f^g \circ \theta$. Since $f^\delta_1 = f^g_1$, we have $\theta_1 = \mathbf{1}_V$. It follows that $\theta : V^\delta \to V^g$ is an isomorphism. □

Remark 1.4.2.2. The proof shows that the morphisms m^δ_i, $i \geq 1$, and f^δ_i, $i \geq 1$, defined in terms of trees indeed define A_∞-structures (see [KS01, 6.4] for another proof).

Remark 1.4.2.3. If A is a graded differential algebra, the A_∞-morphism

$$g : A \to K$$

has only two non-zero components g_1 and g_2. The complexity of the formulas for m^δ_i, $i \geq 1$, thus arises from the complexity of the formulas for f^δ_i, $i \geq 1$, defining the kernel of g in Alg_∞

$$f^\delta : V^\delta \hookrightarrow A.$$

Remark 1.4.2.4. The perturbation lemma gives us, in addition to V^δ and f^δ, a contraction of the A_∞-algebras

$$\Big(V^\delta \underset{f^\delta}{\overset{q^\delta}{\rightleftarrows}} A \ , H^\delta \Big).$$

Note that the A_∞-morphism q^δ is the cokernel of the A_∞-morphism

$$j : K \to A$$

given by the formulas

$$j_1 = \sigma, \quad j_i = -m_i \circ (\sigma^{\otimes i}) \circ (h \otimes \mathbf{1}^{\otimes i-1}), \quad i \geq 2.$$

□

Remark 1.4.2.5. Let V and W be complexes equipped with exhaustive filtrations and let

$$\big(\, (V, d_V) \underset{i}{\overset{\rho}{\rightleftarrows}} (W, d_W)\ , H\big)$$

be a filtered contraction of complexes. Then there exists a split exact sequence of complexes

$$0 \longrightarrow (V, d_V) \underset{i}{\overset{\rho}{\rightleftarrows}} (W, d_W) \underset{p}{\overset{\sigma}{\rightleftarrows}} (K, d_K) \longrightarrow 0$$

such that

$$\rho \circ \sigma = 0 \quad \text{and} \quad i \circ \rho + \sigma \circ p = \mathbf{1}_A$$

and a contracting homotopy h of K such that

$$h^2 = 0 \quad \text{and} \quad H = \sigma \circ h \circ p.$$

The contractible complex K is therefore a direct factor of W. Let δ be a perturbation of the differential d_W. The perturbation lemma produces a filtered contraction of the complexes

$$\big(\, (V, d_V^\delta) \underset{i^\delta}{\overset{\rho^\delta}{\rightleftarrows}} (W, d_W + \delta)\ , H^\delta\big).$$

A calculation shows that the morphisms

$$(p - pH\delta) : (W, d_W + \delta) \to (K, d_K) \quad \text{and} \quad (\sigma - \delta H\sigma) : (K, d_K) \to (W, d_W + \delta)$$

are morphisms of complexes and that they are the cokernel and kernel of i^δ and ρ^δ. The composition

$$(p - pH\delta) \circ (\sigma - \delta H\sigma) : (K, d_K) \to (K, d_K)$$

induces an isomorphism on the graded objects associated to the filtration. Therefore, it is an isomrphism. The contractible complex (K, d_K) is also a direct factor of the perturbed complex $(W, d_W + \delta)$ and the inclusion

$$\sigma : K \to W$$

is "perturbed" to $\sigma - \delta H\sigma$ in order to become compatible with $d_W + \delta$.

Chapter 2

The homotopy theory of polydules

Introduction

Let A be an augmented A_∞-algebra. Recall that in this thesis the structures commonly called A_∞-modules on A are called *A-polydules* ("poly" because the structure is given by several multiplications). The purpose of this chapter is to describe the derived category $\mathcal{D}_\infty A$ whose objects are the strictly unital A-polydules. For this, we will use the tools of Quillen's homotopic algebra (see appendix A) by adapting the methods of chapter 1 to polydules. The derived category of any A_∞-algebra will be studied in chapter 4.

Chapter plan

This chapter is divided into two parts.

The first part, which is made up of sections (2.1) and (2.2) will not deal with the A_∞-structures themselves. In the first section (2.1), we define the (co)unital differential graded (co)modules. In the section 2.2, we prove theorem (2.2.2.2):
Let C be a coaugmented cocomplete differential graded coalgebra. The category $\mathsf{Comc}\, C$ of cocomplete counital differential graded C-comodules admits a unique model category structure such that, for any augmented differential graded algebra A and any admissible acyclic twisting cochain $\tau : C \to A$, the pair of adjoint functors

$$(? \otimes_\tau A, - \otimes_\tau C) : \mathsf{Comc}\, C \to \mathsf{Mod}\, A$$

is a Quillen equivalence. All objects of $\mathsf{Comc}\, C$ are cofibrant.
We then characterize the acyclicity of twisting cochains (Proposition 2.2.4.1).

The second part is devoted to the A_∞-structures concerned in this chapter: strictly unital (bi)polydules on augmented A_∞-algebras. In section 2.3, we define polydules, their suspensions, A_∞-morphisms and homotopies between A_∞-morphisms. We then define the notion of strict unitarity for A_∞-structures. This notion will be studied more precisely in chapter 3. We then recall the bar and cobar constructions and the enveloping algebra. In section 2.4, we refine the aforementioned theorem (2.2.2.2). We show that, if the coalgebra C is isomorphic, as a graded coalgebra, to a co-augmented tensor coalgebra, the fibrant objects of $\mathsf{Comc}\, C$ are exactly the direct factors of the almost cofree C-comodules. In particular, in the case where C is equal to the bar construction of an augmented A_∞-algebra A, the category of fibrant and cofibrant objects of $\mathsf{Comc}\, C$ is the essential image by the bar construction of strictly unital A-polydules. We will deduce from this result several descriptions of the derived category

$$\mathcal{D}_\infty A = \mathsf{Mod}_\infty A[Qis^{-1}],$$

where $\mathsf{Mod}_\infty A$ denotes the category of strictly unital A-polydules.

In section 2.5, we study the derived category $\mathcal{D}_\infty(A, A')$ of strictly unital bipolydules on A and A', two augmented A_∞-algebras. Since the methods are similar, details will be omitted. Bipolydules will be useful in the study of A_∞-categories.

2.1 Reminders and notations

Let $(\mathsf{C}, \otimes, e)$ be a monoidal semi-simple Grothendieck $\mathbb{K}$-category and C' be a semi-simple Grothendieck $\mathbb{K}$-category (not necessarily monoidal). We assume that the monoidal category C acts on the right on C', i.e. C' is endowed with a functor

$$\mathsf{C}' \times \mathsf{C} \to \mathsf{C}', \quad (M, A) \mapsto M \otimes A$$

such that

$$\mathsf{Hom}_{\mathsf{C}'}(M, M') \times \mathsf{Hom}\mathsf{C}(A, A') \to \mathsf{Hom}_{\mathsf{C}'}(M \otimes A, M' \otimes A'),$$

where A, A' are in C and M, M' are in C', is $\mathbb{K}$-bilinear. We further require that this action be associative and unital up to given isomorphisms (see [ML98, chap. XI]).

2.1.1 Modules over an augmented algebra

Let M (resp. M') be one of the categories $\mathcal{G}r\mathsf{C}$ or $\mathcal{C}\mathsf{C}$ (resp. $\mathcal{G}r\mathsf{C}'$ or $\mathcal{C}\mathsf{C}'$) defined in section 1.1.1. The category M is monoidal and clearly acts on M'.

Augmented algebras, reduced algebras

An algebra (A, μ) in M is *unital* if it is equipped with a morphism $\eta : e \to A$ such that $\mu(\mathbf{1} \otimes \eta) = \mu(\eta \otimes \mathbf{1}) = \mathbf{1}$. We call the morphism η the *unit* of A. If A and A' are unital algebras, a *morphism* of unital algebras $f : A \to A'$ is an algebra morphism f such that $f\eta_A = \eta_{A'}$. The morphism $e \otimes e \to e$ given by the unital constraint of the base category (in Section 1.1.1) defines a unital algebra structure on the unit object e. An algebra A is *augmented* if it is unital and equipped with a morphism of unital algebras

$$\varepsilon : A \to e.$$

The morphism ε is called the *augmentation* of A. If A and A' are augmented algebras, a *morphism* of augmented algebras $f : A \to A'$ is a morphism of unital algebras f such that $\varepsilon_{A'} f = \varepsilon_A$.

If A is an augmented algebra in M, the *reduced algebra* $\overline{A}$ associated to A is the kernel of the augmentation. If A is an algebra in M, the *augmented algebra* associated to A is the algebra A^+ whose underlying object is $e \oplus A$, and whose multiplication is defined by the morphisms

$$e \otimes e \to e, \quad e \otimes A \to A, \quad A \otimes e \to A \quad \text{and} \quad A \otimes A \xrightarrow{\mu} A,$$

where the first three morphisms are given by the unital constraint of the base category. The augmentation of A^+ is the canonical projection $A^+ \to e$. We denote by Alga the *category* of augmented algebras of $\mathcal{C}\mathsf{C}$. The functor

$$\mathsf{Alg} \longrightarrow \mathsf{Alga}, \quad A \mapsto A^+,$$

is an equivalence whose quasi-inverse is the functor $A \mapsto \overline{A}$.

Modules

Let A be an algebra in M. A *(right) A-module in* M' is an object M in M' equipped with a morphism $\mu^M : M \otimes A \to M$ (of degree 0 if $\mathsf{M}' = \mathcal{G}r\mathsf{C}'$) such that

$$\mu^M(\mu^M \otimes \mathbf{1}) = \mu^M(\mathbf{1} \otimes \mu^A).$$

We denote by μ^M the *multiplication of M*. If M and N are two modules, a *morphism* of modules $f : M \to N$ is a morphism f such that

$$f\mu^M = \mu^N(f \otimes \mathbf{1}).$$

If the algebra A is unital, an A-module M is *unital* if we have

$$\mu^M(\mathbf{1} \otimes \eta^A) = \mathbf{1}_M.$$

Let A be a graded (resp. differential graded) algebra. A *graded (resp. differential graded) A-module* is an A-module in the category $\mathcal{G}r\mathsf{C}'$ (resp. $\mathcal{C}\mathsf{C}'$). If A is a differential graded algebra, a differential graded A-module is therefore an object M in $\mathcal{G}r\mathsf{C}'$, endowed with a multiplication $\mu^M : M \otimes A \to M$ and with a differential $d^M : M \to M$ such that

$$d^M(\mu^M) = \mu^M(d^M \otimes \mathbf{1}_A + \mathbf{1}_M \otimes d^A).$$

If (M, μ^M) is a graded A-module, a *derivation of modules* is a morphism $d^M : M \to M$ satisfying the above equation. A *module differential* is a derivation of degree $+1$ that squares to zero. If A is a unital differential graded algebra, we denote by $\mathsf{Mod}\, A$ the *category* of unital differential graded A-modules.

Let $f : A \to A'$ be a morphism of Alg. The *restriction along f* of an A'-module M is the A-module whose underlying object is M and whose multiplication is $\mu^M(f \otimes \mathbf{1})$. The *$A'$-module induced by f* of an A-module M has for underlying object $M \otimes_A A'$ and for multiplication $\mathbf{1} \otimes \mu^{A'}$. Let A be an augmented algebra and let $i : \overline{A} \to A$ the canonical injection. The *restriction functor* is an equivalence of $\mathsf{Mod}\, A$ on the category of differential graded modules on $\overline{A}$, its quasi-inverse is the *induction functor*.

Let A be a differential graded algebra and M and N be two differential graded modules. If f and g are two morphisms $M \to N$, a *homotopy* between f and g is a graded morphism of A-modules $h : M \to N$ of degree -1 such that $h \circ d + d \circ h = f - g$. Two morphisms f and g are *homotopic* if there exists a homotopy between f and g.

Free modules

Let A be an algebra in M. Let V be an object in M'. The morphism $\mathbf{1}_V \otimes \mu^A$ defines an A-module structure on $V \otimes A$. An A-module M is *free over* V if there exists an isomorphism of A-modules $M \xrightarrow{\sim} V \otimes A$. A differential graded module is *almost free* if it is free as a graded module.

Lemma 2.1.1.1. Let A be an object in de Alga. Let M be an object in $\mathsf{Mod}\,A$ and V an object in $\mathcal{G}r\mathsf{C}'$.

a. The map $f \mapsto f(\mathbf{1}\otimes\eta)$ is a bijection from the set of morphisms of graded modules $V\otimes A \to M$ to the set of graded morphisms $V \to M$. The inverse map associates to $g : V \to M$ the morphism of modules
$$V \otimes A \xrightarrow{g\otimes\mathbf{1}} M \otimes A \xrightarrow{\mu^M} M.$$

b. The map $d \mapsto d(\mathbf{1} \otimes \eta)$ is a bijection from the set $\mathcal{E}$ of derivations of graded modules $V \otimes A$ to the set of graded module morphisms $g : V \to V \otimes A$. The inverse map associates to $g : M \to N$ the differential
$$\mathbf{1} \otimes d^A + (\mathbf{1} \otimes \mu^A)(g \otimes \mathbf{1}).$$
This bijection maps the subset of $\mathcal{E}$ formed by differentials of modules to morphisms of degree $+1$ such that
$$(\mathbf{1}_V \otimes \mu^A)(g \otimes \mathbf{1})g + (\mathbf{1} \otimes d^A)g = 0.$$

□

2.1.2 Coaugmented comodules

Coaugmented coalgebras, reduced coalgebras

A coalgebra (C, Δ) of M is *co-unital* if it is endowed with a morphism $\eta : C \to e$ such that $(\mathbf{1}\otimes\eta)\Delta = (\eta\otimes\mathbf{1})\Delta = \mathbf{1}$. The morphism η is called the *co-unit* of C. If C and C' are two co-unital coalgebras, a *morphism* of co-unital coalgebras $f : C \to C'$ is a morphism of coalgebras f such that $\eta_{C'}f = \eta_C$. The morphism $e \to e \otimes e$ given by the unital constraint of the base category defines a co-unital coalgebra structure on the neutral object e. A coalgebra C is *co-augmented* if it is endowed with a morphism of co-unital coalgebras
$$\varepsilon : e \to C.$$
The morphism ε is called the *co-augmentation* of the coalgebra C. If C and C' are two co-augmented coalgebras, a *morphism* of co-augmented coalgebras $f : C \to C'$ is a morphism of unital coalgebras f such that $f\varepsilon_C = \varepsilon_{C'}$.

If C is a coaugmented coalgebra in M, the *reduced coalgebra* $\overline{C}$ is the cokernel of the co-augmentation. If C is a coalgebra in M, the *co-augmented coalgebra* C^+ is the coalgebra whose underlying object is $C \oplus e$ and whose comultiplication is the morphism defined by the components
$$e \to e \otimes e, \quad C \to e \otimes C, \quad C \to C \otimes e \quad \text{and} \quad C \xrightarrow{\Delta} C \otimes C,$$
where the first three morphisms are defined by the unital constraint of the base category. The co-augmentation of C^+ is the canonical injection $e \to C^+$. If V is a graded object of C, we denote

by T^cV the coalgebra $(\overline{T^c}V)^+$. Let Cogca be the *category* of co-augmented coalgebras of $\mathcal{C}\mathsf{C}$ whose reduced coalgebras are cocomplete. The functor

$$\mathsf{Cogc} \to \mathsf{Cogca}, \quad C \mapsto C^+,$$

is an equivalence whose quasi-inverse is the functor $C \to \overline{C}$.

Comodules

Let C be a coalgebra of M. A C-*(right) comodule in* M' is a graded object N of M' endowed with a morphism $\Delta^N : N \to N \otimes C$ (of degree 0 if $\mathsf{M}' = \mathcal{G}r\mathsf{C}'$) such that

$$(\mathbf{1} \otimes \Delta^C)\Delta^N = (\Delta^N \otimes \mathbf{1})\Delta^N.$$

If N and N' are two C-comodules, a *morphism* of C-comodules $f : N \to N'$ is a morphism of M' such that $\Delta^{N'} f = (f \otimes \mathbf{1})\Delta^N$. If a coalgebra C is co-unital, a C-comodule N is *co-unital* if $\Delta^N(\mathbf{1} \otimes \eta) = \mathbf{1}_N$.

Let C be a graded (resp. differential graded) coalgebra. A C-*graded comodule* (resp. *differential graded comodule*) is a C-comodule in the category $\mathcal{G}r\mathsf{C}'$ (resp. $\mathcal{C}\mathsf{C}'$). If C is a differential graded coalgebra, a differential graded comodule is therefore an object N of $\mathcal{G}r\mathsf{C}'$, endowed with a comultiplication $\Delta^N : N \to N \otimes C$ and a differential $d^N : N \to N$ such that

$$\Delta^N d^N = (d^N \otimes \mathbf{1}_A + \mathbf{1}_N \otimes d^A)\Delta^N.$$

If (N, Δ^N) is a graded C-comodule, a *coderivation of comodules* is a morphism $d^N : N \to N$ satisfying the above equation. A *comodule differential* is a coderivation of degree $+1$ that squares to zero. If the coalgebra C is co-unital, we denote by $\mathsf{Com}\, C$ the category of co-unital differential graded comodules.

Let $f : C \to C'$ be a morphism of Cog. The *corestriction along* f of a C-comodule N is the C'-comodule whose underlying object is N and whose comultiplication is $(\mathbf{1} \otimes f)\Delta^N$. The C-*comodule co-induced by* f associated to a C'-comodule N has as its underlying object the kernel

$$\ker(N \otimes C \xrightarrow{u} N \otimes C' \otimes C),$$

where $u = \Delta^N \otimes \mathbf{1}_C - (\mathbf{1}_N \otimes f \otimes \mathbf{1}_C)(\mathbf{1}_N \otimes \Delta^C)$, and for comultiplication the morphism induced by $\mathbf{1}^N \otimes \Delta^C : N \otimes C \to N \otimes C \otimes C$.

Let C be a co-augmented coalgebra and let $p : C \to \overline{C}$ be the canonical projection. The *corestriction functor* is an equivalence of the category $\mathsf{Com}\, C$ over the category of differential graded $\overline{C}$-comodules. Its quasi-inverse is the *co-induction functor*.

Let C be a differential graded coalgebra and let N and N' be two differential graded comodules. If f and g are two morphisms $N \to N'$, a *homotopy* between f and g is a graded morphism of C-comodules $h : N \to N'$ of degree -1 such that $h \circ d + d \circ h = f - g$. Two morphisms f and g are *homotopic* if there is a homotopy between f and g.

Cocomplete comodules

Let C be a co-augmented coalgebra of M and N a co-unital C-comodule in M'. We define $\Delta^{(2)} = \Delta^N$ and, for all $n \geq 3$, we define $\Delta^{(n)} : N \to N \otimes C^{\otimes n-1}$ by

$$\Delta^{(n)} = (\mathbf{1}^{\otimes n-2} \otimes \Delta^C)\Delta^{(n-1)}.$$

Let $n \geq 1$. The kernel $N_{[n]}$ of the morphism

$$N \xrightarrow{\Delta^{(n+1)}} N \otimes C^{\otimes n} \xrightarrow{\mathbf{1}\otimes p^{\otimes n}} N \otimes \overline{C}^{\otimes n}$$

(where $p : C \to \overline{C}$ is the canonical projection) is a sub-comodule of N. It is called the *sub-comodule of n-primitives* of N. For $n = 1$, we obtain the sub-comodule of *primitives* of N. The increasing sequence of sub-comodules

$$N_{[1]} \subset N_{[2]} \subset N_{[3]} \subset \cdots$$

is the *primitive filtration* of the comodule N. If C is an object of Cogca, a co-unital differential graded C-comodule N is *cocomplete* if its primitive filtration is exhaustive. We denote by $\mathsf{Comc}\, C$ the category of cocomplete comodules.

Cofree comodules

Let C be a co-augmented coalgebra in M. Let V be an object of M'. The morphism

$$\mathbf{1} \otimes \Delta^C : V \otimes C \to V \otimes C \otimes C$$

endows $V \otimes C$ with a C-comodule structure. Its sub-comodule of primitives is the comodule $V \otimes e$. For $n \geq 2$, its sub-comodule of n-primitives is the C-comodule $V \otimes C_{[n-1]}$. The C-comodule $V \otimes C$ is therefore cocomplete if C is an object of Cogca. A C-module N is *cofree over V* if there exists an isomorphism of C-comodules $N \xrightarrow{\sim} V \otimes C$. If C is an object of Cogca, a differential graded comodule is *almost cofree* if it is free as a graded comodule. The *sub-category* of $\mathsf{Comc}\, C$ consisting of almost cofree objects is denoted by $\mathsf{prcol}\, C$.

Lemma 2.1.2.1. Let C be a co-unital differential graded coalgebra, N an object in $\mathsf{Com}\, C$ and V a graded object.

a. The map $f \mapsto (\mathbf{1}\otimes\eta^C)f$ is a bijection from the set of graded comodule morphisms $N \to V \otimes C$ to the set of graded morphisms $N \to V$. The inverse map sends $g : N \to V$ to the morphism of C-comodules

$$N \xrightarrow{\Delta} N \otimes C \xrightarrow{g\otimes\mathbf{1}} V \otimes C.$$

b. The map $d \mapsto (\mathbf{1} \otimes \eta^C)d$ is a bijection from the set

$$\mathsf{coder}(V \otimes C)$$

of coderivations of comodules $V \otimes C$ to the set of graded morphisms $g : V \otimes C \to V$. The inverse map sends g to the co-derivation

$$(g \otimes \mathbf{1})(\mathbf{1}_V \otimes \Delta^C) + \mathbf{1}_V \otimes d^C.$$

This bijection maps comodule differentials to graded morphisms of degree $+1$ such that

$$g(\mathbf{1}_V \otimes d^C) + g(g \otimes \mathbf{1}_C)(\mathbf{1}_V \otimes \Delta^C) = 0.$$

□

2.2 $\mathsf{Comc}\, C$ as a model category

2.2.1 Twisting cochain and twisted tensor products

Definition 2.2.1.1. Let C be a differential graded coalgebra and A a differential graded algebra. A *twisting cochain* is a graded morphism $\tau : C \to A$ of degree $+1$ such that

$$d_A\tau + \tau d_C + m(\tau \otimes \tau)\Delta = 0.$$

If $f : A \to A'$ is a morphism in Alg (resp. if $g : C' \to C$ is a morphism in Cog) the composition $f \circ \tau$ (resp. $\tau \circ g$) is again a twisting cochain. Thus, a twisting cochain $\tau : C \to A$ induces a twisting cochain $\tau^+ = i \circ \tau \circ p : C^+ \to A^+$, where i is the canonical injection $A \to A^+$ and p the canonical projection $C^+ \to C$. Let A be an object of Alga and C an object of Coga. A twisting cochain $C \to A$ is *admissible* if it is induced by a twisting cochain $\overline{C} \to \overline{A}$.

Let A be an augmented differential graded algebra and C a co-augmented differential graded coalgebra. Let $\tau : C \to A$ be an admissible twisting cochain. Let M be an object of $\mathsf{Mod}\, A$. Consider the morphism $t_\tau : M \otimes C \to M \otimes C$ defined as the composition

$$M \otimes C \xrightarrow{\mathbf{1}\otimes\Delta} M \otimes C \otimes C \xrightarrow{\mathbf{1}\otimes\tau\otimes\mathbf{1}} M \otimes A \otimes C \xrightarrow{\mu^M\otimes\mathbf{1}} M \otimes C.$$

Since τ is a twisting cochain, the sum

$$b_\tau = b + t_\tau : M \otimes C \longrightarrow M \otimes C$$

where b is the differential of the tensor product $M \otimes C$, gives a differential on the co-unital graded C-comodule $M \otimes C$. The tensor product $M \otimes C$ endowed with the *twisted (by τ) differential* b_τ is denoted $M \otimes_\tau C$. If M and M' are two objects of $\mathsf{Mod}\, A$, a morphism $f : M \to M'$ induces a morphism of counital graded C-comodules $f \otimes \mathbf{1}_C : M \otimes_\tau C \to M' \otimes_\tau C$ compatible with differentials. We thus obtain a *functor*

$$R_\tau : \mathsf{Mod}\, A \to \mathsf{Com}\, C, \qquad M \mapsto M \otimes_\tau C.$$

When there is no ambiguity we will denote this functor by R.

Dually, if N is a differential graded co-unital C-comodule, the *morphism* T_τ is defined as the composition

$$N \otimes A \xrightarrow{\Delta^N\otimes\mathbf{1}} N \otimes C \otimes A \xrightarrow{\mathbf{1}\otimes\tau\otimes\mathbf{1}} N \otimes A \otimes A \xrightarrow{\mathbf{1}\otimes\mu^A} N \otimes C.$$

The sum of the differential D of the tensor product $N \otimes A$ and of the morphism T_τ defines a new differential on the unital graded A-module $N \otimes A$. The tensor product $N \otimes A$ endowed with the *twisted (by τ) differential* $D_\tau = D + T_\tau$ is denoted by $N \otimes_\tau A$. If N and N' are two objects of $\mathsf{Comc}\, C$, a morphism $f : N \to N'$ induces a morphism of unital graded A-modules $f \otimes \mathbf{1}_A : N \otimes_\tau A \to N' \otimes_\tau A$ compatible with differentials. We thus obtain a *functor*

$$L_\tau : \mathsf{Com}\, C \to \mathsf{Mod}\, A, \qquad N \to N \otimes_\tau A$$

that we will denote as L when there is no ambiguity.

Lemma 2.2.1.2. The functor $L : \mathsf{Com}\, C \to \mathsf{Mod}\, A$ is left adjoint to the functor $R : \mathsf{Mod}\, A \to \mathsf{Com}\, C$.

Proof. Let N be an object of $\mathsf{Com}\,C$ and M an object of $\mathsf{Mod}\,A$. We give the functorial bijection

$$\phi : \mathsf{Hom}_{\mathsf{Mod}\,A}(LN, M) \longrightarrow \mathsf{Hom}_{\mathsf{Com}\,C}(N, RM).$$

Let $f : LN \to M$ be a morphism from $\mathsf{Mod}\,A$. By Lemma 2.1.1.1, it is determined by its composition $\alpha = f \circ (\mathbf{1}_N \otimes \eta^A) : N \to M$. By Lemma 2.1.2.1, the morphism α in turn determines a graded morphism of co-unital C-comodules $\phi(f) : N \to RM$ such that $(\mathbf{1} \otimes \eta^C)\phi(f) = \alpha$. We verify that the condition $b_\tau \phi(f) - \phi(f) d_N = 0$ is equivalent to the condition $d_M f - f D_\tau = 0$. □

Definition 2.2.1.3. An admissible twisting cochain $\tau : C \to A$ is *acyclic* if, for any object M of $\mathsf{Mod}\,A$, the adjunction morphism

$$\phi : LRM \to M$$

is a quasi-isomorphism (see Proposition 2.2.4.1 below for equivalent conditions).

Notation 2.2.1.4 (Bar and cobar construction)**.** Let A be an object of Alga. We denote by B^+A the co-augmented coalgebra $(B\overline{A})^+$, where $\overline{A}$ is the reduced algebra associated to A. Be careful not to confuse the co-augmented cogebras B^+A and $(BA)^+$. Let C be an object of Cogca. We denote by Ω^+C the augmented algebra $(\Omega\overline{C})^+$, where $\overline{C}$ is the reduced coalgebra associated to C. It is not isomorphic to $(\Omega C)^+$.

Lemma 2.2.1.5.

a. Let A be an object of Alga. Let $p : B\overline{A} \to S\overline{A}$ be the canonical projection. The composition

$$\tau_A : B^+A \to B\overline{A} \xrightarrow{\omega \circ p} \overline{A} \to A,$$

where the first arrow is the canonical projection and the last is the canonical injection, is an admissible twisting cochain. The cochain τ_A is universal among the admissible twisting cochains of target A, i.e. if C is an object of Coga and $\tau : C \to A$ is an admissible twisting cochain, there exists a unique morphism g_τ such that $\tau_A \circ g_\tau = \tau$.

b. In a dual way, we associate to an object C of Cogca an admissible twisting cochain

$$\tau_C : C \to \overline{C} \xrightarrow{i \circ \omega} \Omega\overline{C} \to \Omega^+C$$

where $i : S^{-1}\overline{C} \to \Omega\overline{C}$ is the canonical injection. The cochain τ_C is universal among the admissible twisting cochains of source C, i.e. if $\tau : C \to A$ is an admissible twisting cochain, there exists a unique morphism f_τ such that $f_\tau \circ \tau_C = \tau$.

Proof. Let C be an object of Cogca and A an object of Alga. Let $\tau : C \to A$ be a graded morphism of degree $+1$ whose composition with the co-augmentation of C and the augmentation of A is zero. That is

$$f_\tau : \Omega^+C \to A$$

the graded morphism of augmented algebras lifts (Lemma 1.1.2.1) the composition $\tau \circ s$ and

$$g_\tau : C \to B^+A$$

the graded morphism of co-augmented coalgebras lifts (Lemma 1.1.2.2) the composition $s \circ \tau$. By the proof of Lemma 1.2.2.5, the graded morphism τ is a twisting cochain if and only if f_τ is compatible with differentials if and only if g_τ is compatible with differentials.

a. The composition $\omega \circ p : B\overline{A} \to \overline{A}$ is a twisting cochain because the lifting (Lemma 1.1.2.2) of $p : B\overline{A} \to S\overline{A}$ is the identity of the coalgebra $B\overline{A}$ (and the latter obviously commutes with the differential of $B\overline{A}$). Universality is immediate.

b. Idem. □

Definition 2.2.1.6. We call τ_A the *universal twisting cochain of* A and τ_C the *universal twisting cochain of* C.

Remark 2.2.1.7. In [HMS74], the functor

$$R_{\tau_A} : \mathsf{Mod}\, A \to \mathsf{Comc}\, B^+A, \qquad M \mapsto M \otimes_{\tau_A} B^+A,$$

is denoted $B_A M$.

Denote by

$$\mathsf{Res} : \mathsf{Mod}\, A \to \mathsf{Mod}\, \Omega^+C$$

the restriction functor along f_τ and by

$$\mathsf{Ind} : \mathsf{Mod}\, \Omega^+C \to \mathsf{Mod}\, A$$

the induction functor. We know that $(\mathsf{Ind}, \mathsf{Res})$ is a pair of adjoint functors from the category $\mathsf{Mod}\, \Omega^+C$ to the category $\mathsf{Mod}\, A$. Denote by

$$\mathsf{Res}^{op} : \mathsf{Comc}\, C \to \mathsf{Comc}\, B^+A$$

the corestriction functor along g_τ and by

$$\mathsf{Ind}^{op} : \mathsf{Comc}\, B^+A \to \mathsf{Comc}\, C$$

the co-induction functor. We know that $(\mathsf{Res}^{op}, \mathsf{Ind}^{op})$ is a pair of adjoint functors from the category $\mathsf{Comc}\, C$ to the category $\mathsf{Comc}\, B^+A$.

Lemma 2.2.1.8.

a. The pair of adjoint functors (L_τ, R_τ) from the category $\mathsf{Mod}\, A$ to the category $\mathsf{Comc}\, C$ is the composition of the pair $(\mathsf{Ind}, \mathsf{Res})$ with the pair (L_{τ_C}, R_{τ_C}).

b. The pair of adjoint functors (L_τ, R_τ) from the category $\mathsf{Mod}\, A$ to the category $\mathsf{Comc}\, C$ is the composition of the pair (L_{τ_A}, R_{τ_A}) with the pair $(\mathsf{Res}^{op}, \mathsf{Ind}^{op})$.

□

Lemma 2.2.1.9.

a. Let A be an object of Alga. The universal twisting cochain τ_A is acyclic.

b. Let C be an object of Cogca. The universal twisting cochain τ_C is acyclic.

Proof.
a. Let M be an object of $\mathsf{Mod}\,A$. Let us show that $LRM = (M \otimes B^{+}A \otimes A, d)$ is a resolution (known as the normalized bar resolution) of M

$$bar_A(M) = \cdots \to M \otimes \overline{A}^{\otimes i} \otimes A \to \cdots \to M \otimes \overline{A} \otimes A \to M \otimes A,$$

and that the morphism Φ corresponding to the morphism $bar_A(M) \to M$ is a quasi-isomorphism. As in the case where M is concentrated in degree 0, (see [CE99, IX.6] where this complex is called the normalized standard complex) the morphisms

$$h_{i-1} = \mathbf{1}^{\otimes i} \otimes p \otimes \varepsilon : M \otimes \overline{A}^{\otimes i-1} \otimes A \to M \otimes \overline{A}^{\otimes i} \otimes A,$$

where p is the canonical projection, define a contracting homotopy of the complex

$$\cdots \to M \otimes \overline{A}^{\otimes i} \otimes A \to \cdots \to M \otimes \overline{A} \otimes A \to M \otimes A \to M \to 0.$$

b. Let M be an object of $\mathsf{Mod}\,\Omega^{+}C$. Let us show that $\Phi : LRM \to M$ is a filtered quasi-isomorphism. We endow $\Omega\overline{C}$ with the filtration induced by the primitive filtration of $\overline{C}$ considered as a coalgebra. We then have a filtration of $\Omega^{+}C$ defined by

$$\left(\Omega^{+}C\right)_i = (\Omega\overline{C})_i \oplus e, \quad i \geq 0.$$

Equip C, considered as an object of $\mathsf{Com}\,C$, with its primitive filtration as a C-module (we complete it with $C_{[0]} = e$). Equip M with the filtration defined by $M_i = M$, $i \geq 0$. These filtrations induce on $LRM = (M \otimes C \otimes \Omega^{+}C)$ a filtration of complexes. The morphism $\Phi : LRM \to M$ becomes a filtered morphism for these filtrations. It induces a morphism

$$\mathsf{Gr}_0(LRM) \to \mathsf{Gr}_0 M$$

which is the identity of M. Since $\mathsf{Gr}_i M = 0$ for all $i \geq 1$, it suffices to show that

$$\mathsf{Gr}_i(LRM), \quad i \geq 1,$$

is contractible. Let $i \geq 1$. By construction, we have an isomorphism of graded objects

$$\mathsf{Gr}_i(LRM) = M \otimes e \otimes \mathsf{Gr}_i\Omega^{+}C \oplus \Big(\bigoplus_{\substack{i_1+i_2=i \\ i_1 \neq 0}} M \otimes \mathsf{Gr}_{i_1} C \otimes \mathsf{Gr}_{i_2}\Omega^{+}C \Big).$$

The differential has as a matrix

$$\begin{bmatrix} 0 & \rho \\ 0 & 0 \end{bmatrix}$$

where ρ is the morphism induced by T_{τ_C}

$$\bigoplus_{\substack{i_1+i_2=i \\ i_1 \neq 0}} M \otimes \mathsf{Gr}_{i_1} C \otimes \mathsf{Gr}_{i_2}\Omega^{+}C \longrightarrow M \otimes e \otimes \mathsf{Gr}_i\Omega^{+}C.$$

The latter is an isomorphism because it is induced by the isomorphism

$$\bigoplus_{\substack{i_1+i_2=i \\ i_1 \neq 0}} \mathsf{Gr}_{i_1} C \otimes \mathsf{Gr}_{i_2}\Omega^{+}C \longrightarrow \mathsf{Gr}_i\Omega^{+}C$$

□

2.2.2 $\mathsf{Comc}\, C$ as a model category

Let C be an object of Cogca.

In this section, we will equip $\mathsf{Comc}\, C$ with a model category structure. We start by recalling the model category structure on $\mathsf{Mod}\, A$, where A is an object of Alga and we then state the main theorem (2.2.2.2). We will not detail all of its proof because it is similar to that of (Theorem 1.3.1.2). Only points that differ will be developed.

Reminders on the category $\mathsf{Mod}\, A$

Let A be a unital differential graded algebra. In the category $\mathsf{Mod}\, A$, consider the following three classes of morphisms

- the class Qis of quasi-isomorphisms,
- the class $\mathcal{F}ib$ of morphisms $f : M \to M'$ such that f^n is an epimorphism for all $n \in \mathbf{Z}$,
- the class $\mathcal{C}of$ of morphisms which have the left-lifting-property with respect to the morphisms belonging to $Qis \cap \mathcal{F}ib$.

Theorem 2.2.2.1 (Hinich [Hin97])**.** The category $\mathsf{Mod}\, A$ equipped with the classes of morphisms defined above is a model category. All objects are fibrant. The cofibrant objects are described in Remark 2.2.2.10 below.

The principal theorem

Let A be an object of Alga and C an object of Cogca. Let $\tau : C \to A$ be an acyclic admissible twisting cochain. In the category $\mathsf{Comc}\, C$ of cocomplete counital differential graded comodules, we consider the following three classes of morphisms:

- the class $\mathcal{E}q$ of *weak equivalences* is formed of the morphisms $f : N \to N'$ such that $Lf : LN \to LN'$ is a quasi-isomorphism of modules,
- the class $\mathcal{C}of$ of *cofibrations* is made up of the morphisms $f : N \to N'$ which, as morphisms of complexes, are monomorphisms,
- the class $\mathcal{F}ib$ of *fibrations* is made up of morphisms which have the right-lifting-property with respect to trivial cofibrations.

Theorem 2.2.2.2.

a. The category $\mathsf{Comc}\, C$ equipped with the three classes of morphisms above is a model category. All its objects are cofibrant. An object of $\mathsf{Comc}\, C$ is fibrant if and only if it is a direct factor of an object RM, where M is an object of $\mathsf{Mod}\, A$.

b. Equip the category $\mathsf{Mod}\, A$ with the model category structure of Theorem 2.2.2.1. The pair of adjoint functors (L, R) from $\mathsf{Comc}\, C$ to $\mathsf{Mod}\, A$ is a Quillen equivalence.

c. The model category structure on $\mathsf{Comc}\, C$ does not depend on the acyclic admissible twisting cochain τ.

In particular, the category $\mathsf{Ho}\,\mathsf{Comc}\,C$ is equivalent to the derived category $\mathcal{D}A$ (see the definition in 2.2.3). Theorem 2.2.2.2 and Lemma 2.2.1.9 imply the following corollary:

Corollary 2.2.2.3. The category $\mathsf{Comc}\,C$ admits a unique model category structure such that for any admissible acyclic twisting cochain $\tau : C \to A$, where A is an object of Alga, the pair of adjoint functors (L, R) is a Quillen equivalence. □

Definition 2.2.2.4. We call the model category structure on $\mathsf{Comc}\,C$ of the corollary the *canonical structure.*

To prove Theorem 2.2.2.2, we need (like in the proof of Theorem 1.3.1.2) to introduce filtrations.

If the algebra A (resp. coalgebra C) is filtered, a *filtered differential graded A-module* (resp. *filtered differential graded C-comodule*) is an A-module (resp. C-comodule) in the category of filtered complexes. A filtered C-comodule M is *admissible* if its filtration is exhaustive and if $M_0 = 0$. By definition, all objects of $\mathsf{Comc}\,C$, provided with their primitive filtration are admissible.

Lemma 2.2.2.5. If C is endowed with an exhaustive filtration of coalgebras such that $C_0 = e$, a filtered quasi-isomorphism of admissible C-comodules is a weak equivalence.

Proof. Let $f : N \to N'$ be a filtered quasi-isomorphism of admissible C-comodules. The filtration of N induces a filtration of the A-module defined by the sequence

$$(LN)_i = N_i \otimes A, \quad i \geq 0.$$

The differential of $(LN)_i$, $i \geq 0$, is the sum of the differential of the tensor product $N_i \otimes A$ and the contribution from D_τ. Since the filtration of N is admissible and the cochain $\tau : C \to A$ is admissible, the contribution from D_τ decreases the filtration of LN. Thus, the differential of

$$\mathsf{Gr}LN \xrightarrow{\sim} \mathsf{Gr}N \otimes A$$

is that of the tensor product $\mathsf{Gr}N \otimes A$ and the morphism Lf is indeed a quasi-isomorphism of A-modules. □

Lemma 2.2.2.6.

a. Let M and M' be two objects of $\mathsf{Mod}\,A$. The functor R sends a quasi-isomorphism $M \to M'$ to a weak equivalence $Rf : RM \to RM'$ in $\mathsf{Comc}\,C$.

b. Let M be an object of $\mathsf{Mod}\,A$. The adjunction morphism

$$\Phi : LRM \longrightarrow M$$

is a quasi-isomorphism of A-modules.

c. Let N be an object of $\mathsf{Comc}\,C$. The adjunction morphism

$$\Psi : N \longrightarrow RLN$$

is a weak equivalence of $\mathsf{Comc}\,C$.

Proof.
b. The cochain τ is acyclic.

a. The morphism RF is a weak equivalence if and only if LRf is a quasi-isomorphism. By point *b*, Φ is a quasi-isomorphism. Moreover, we have

$$\Phi_M \circ f = LRf \circ \Phi_{M'}.$$

The saturation of quasi-isomorphisms in $\mathsf{Mod}\, A$ gives us the result.

c. We want to show that Ψ is a weak equivalence, that is, $L\Psi : LN \to LRLN$ is a quasi-isomorphism. We know that

$$\Phi_{LN} \circ L\Psi_N = \mathbf{1}_{LN}$$

and that Φ is a quasi-isomorphism. The morphism $L\Psi$ is therefore also a quasi-isomorphism. □

Let us recall the description of [Hin97] of the cofibrations of $\mathsf{Mod}\, A$. The *standard cofibrations* (resp. *trivial cofibrations*) of $\mathsf{Mod}\, A$ are defined as in Definition 1.3.2.5, except that $M^\sharp$ denotes the underlying complex of an object M of $\mathsf{Mod}\, A$ and that FV denotes the differential graded free module on a complex V. We then have the same description (see just below 1.3.2.5) of cofibrations (resp. trivial cofibrations) in $\mathsf{Mod}\, A$ based on the standard cofibrations (resp. trivial cofibrations).

Lemma 2.2.2.7. Let N be an object of $\mathsf{Comc}\, C$ and N' a sub-object of N such that $\Delta N \subset N \otimes e \oplus N' \otimes C$. The functor L sends the inclusion $N' \hookrightarrow N$ to a standard cofibration.

Proof. Let E be the cokernel of the inclusion $N' \hookrightarrow N$. Choose a splitting in the category of graded objects

$$N \xrightarrow{\sim} N' \oplus E.$$

According to this decomposition, the comultiplication Δ^N is given by two components

$$\Delta^{N'} : N' \to N' \otimes C \quad \text{and} \quad \Delta^E = \begin{bmatrix} \Delta_1^E \\ \Delta_2^E \end{bmatrix} : E \longrightarrow N \otimes e \oplus N' \otimes C,$$

and the differential is given by the differential of N', that of E and a morphism

$$d' : E \longrightarrow N'.$$

We have an isomorphism of graded objects

$$LN \xrightarrow{\sim} LN' \oplus LE.$$

The differential is the sum of the differential of $LN' \oplus LE$, the morphism

$$d' \otimes \mathbf{1} : E \otimes A \to N' \otimes A$$

and the morphism d'_τ which is the composition

$$E \otimes A \xrightarrow{\Delta_2^E \otimes \mathbf{1}} N' \otimes C \otimes A \xrightarrow{\mathbf{1} \otimes \tau \otimes \mathbf{1}} N' \otimes A \otimes A \xrightarrow{\mathbf{1} \otimes \mu^A} N' \otimes A.$$

Note that there is no contribution from Δ_1^E because the cochain τ is admissible. Set

$$D' = (d' \otimes \mathbf{1} + d'_\tau)s : S^{-1}E \to N' \otimes A.$$

We verify that LN is isomorphic to

$$LN'\langle S^{-1}E, D'\rangle.$$

□

Lemma 2.2.2.8.

a. The functor L preserves cofibrations and weak equivalences.

b. The functor R preserves fibrations and weak equivalences.

Proof. a. Let $j : N' \rightarrowtail N$ be a cofibration of $\mathsf{Comc}\, C$. Let a filtration of N be given by the sequence

$$N_i = j(N') + N_{[i]}, \quad i \geq 0,$$

where $N_{[i]}$, $i \geq 1$, is the primitive filtration of N (completed by $N_0 = 0$). Note that, for all $i \geq 1$, we have

$$\Delta N_i \subset N_i \otimes e \oplus N_{i-1} \otimes C.$$

We can therefore apply Lemma 2.2.2.7. It gurantees that $LN_i \to LN_{i+1}$ is a standard cofibration. The morphism $Lj : LN' \to LN$ is thus the countable composition of standard cofibrations $LN_i \to LN_{i+1}$, making it a cofibration. By the definition of weak equivalences in $\mathsf{Comc}\, C$, the functor L preserves weak equivalences.

b. By point a and the adjunction (L, R, ϕ) of $\mathsf{Comc}\, C$ in $\mathsf{Mod}\, A$, the functor R preserves fibrations. The fact that it preserves weak equivalences is point a of Lemma 2.2.2.6. □

Lemma 2.2.2.9. Let M be an object of $\mathsf{Mod}\, A$ and N an object of $\mathsf{Comc}\, C$. Consider a fibration $p : M \twoheadrightarrow LN$ of $\mathsf{Mod}\, A$. The morphism $j : RM \prod_{RLN} N \to RM$ of comodules of the cartesian diagram

$$\begin{tikzcd} RM \prod_{RLN} N \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "cart."] & N \arrow[d, "\Psi"] \\ RM \arrow[r, "Rp"'] & RLN. \end{tikzcd}$$

is a trivial cofibration of $\mathsf{Comc}\, C$.

Proof. Let K be the kernel of p. We have isomorphisms of graded objects

$$RM \xrightarrow{\sim} RK \oplus RLN, \quad RM \prod_{RLN} N \xrightarrow{\sim} RK \oplus N.$$

The morphism j is then written

$$\begin{bmatrix} \mathbf{1} & * \\ 0 & \Psi \end{bmatrix}.$$

So we have a diagram of $\mathsf{Mod}\, A$

$$\begin{tikzcd} 0 \arrow[r] & LRK \arrow[r] \arrow[d, "L\mathbf{1}"] & L\Big(RM \prod_{RLN} N\Big) \arrow[r] \arrow[d, "Lj"] & LN \arrow[r] \arrow[d, "L\Psi"] & 0 \\ 0 \arrow[r] & LRK \arrow[r] & RM \arrow[r] & LRLN \arrow[r] & 0, \end{tikzcd}$$

where the lines are exact and where the vertical arrow on the right and the one on the left are quasi-isomorphisms. The morphism Lj is therefore a quasi-isomorphism, and j is a weak equivalence of $\mathsf{Comc}\, C$. It is clearly a monomorphism, therefore a cofibration of $\mathsf{Comc}\, C$. □

Proof of Theorem 2.2.2.2

By the lemmas above, the proof that the classes $\mathcal{E}q$, $\mathcal{C}of$ and $\mathcal{F}ib$ define a model category structure is the same as that of Theorem 1.3.1.2.

Fibrant and cofibrant objects of $\mathsf{Comc}\, C$

All the objects of $\mathsf{Comc}\, C$ are cofibrant since the cofibrations are the monomorphisms.

Let us show that an object of $\mathsf{Comc}\, C$ is fibrant if and only if it is a direct factor of an object RM, where M is an object of $\mathsf{Mod}\, A$. We recall (Theorem 2.2.2.1) that all objects of $\mathsf{Mod}\, A$ are fibrant. By Lemma 2.2.2.8, the image of the functor R is thus formed of fibrant objects of $\mathsf{Comc}\, C$. Thus, all objects of the form RM and their direct factors are fibrant. Conversely if N is fibrant, by the axiom (CM4), the morphism $\Psi : N \to RLN$ (which is a trivial cofibration) is split. The object N is therefore a direct factor of RLN

Remark 2.2.2.10. The dualisation of this proof shows that the cofibrant objects of $\mathsf{Mod}\, A$ are the direct factors of the LN, $N \in \mathsf{Comc}\, C$.

Point b of Theorem 2.2.2.2 is a corollary of Lemma 2.2.2.5. It remains for us to show point c.

Uniqueness of the model category structure on $\mathsf{Comc}\, C$

Let A' be an object of Alga. Let $\tau' : A' \to C$ be an admissible acyclic twisting cochain. We want to show that the model category structure on $\mathsf{Comc}\, C$ (defined at point a of Theorem 2.2.2.2) relative to τ is the same as that relative to τ'.

It suffices to show it in the case where τ' is the universal cochain τ_C. We will show that the classes of cofibrations and the classes of weak equivalences relative to the two structures coincide. This is true for cofibrations since they are monomorphisms. We recall (Lemma 2.2.1.8) that the pair of adjoint functors (L_τ, R_τ) from $\mathsf{Mod}\, A$ to $\mathsf{Comc}\, C$ is the composition of the pair $(\mathsf{Ind}, \mathsf{Res})$ with the pair (L_{τ_C}, R_{τ_C}). As the functor Res induces an equivalence between the localizations of $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, \Omega^+ C$ with respect to quasi-isomorphisms (see [Kel94a, exple 6.1]), the weak equivalences of two structures on $\mathsf{Comc}\, C$ coincide by point b of Theorem 2.2.2.2. □

Filtered quasi-isomorphisms and weak equivalences

We denote by $Qisf$ the class of morphisms $f : N \to N'$ such that C admits an exhaustive coalgebra filtration such that $C_0 = e$ and such that N and N' admit admissible filtrations of C-comodules for which f is a filtered quasi-isomorphism. Lemma 2.2.2.5 shows that we have an inclusion

$$Qisf \subset \mathcal{E}q.$$

We recall (see Appendix A) that the *homotopy category* $\mathsf{Ho}\, \mathsf{Comc}\, C$ is the localization

$$\Big(\mathsf{Comc}\, C\Big)[\mathcal{E}q^{-1}].$$

Lemma 2.2.2.11. The canonical functor

$$\Big(\mathsf{Comc}\, C\Big)[Qisf^{-1}] \xrightarrow{\sim} \mathsf{Ho}\,\mathsf{Comc}\, C$$

is an equivalence.

Proof. The proof is similar to that of point a of Proposition 1.3.5.1. We verify that the adjunction morphism

$$\Psi : N \to R_{\tau_C} L_{\tau_C} N$$

is a filtered quasi-isomorphism morphism for the primitive filtration on N and the filtration on $R_{\tau_C} L_{\tau_C} N$ induced by the primitive filtrations of N and C. The morphism $R_{\tau_C} L_{\tau_C} f$ is clearly a filtered quasi-isomorphism. The saturation property of filtered quasi-isomorphisms applied to the equality $RLf \circ \Psi_N = \Psi_{N'} \circ f$ gives us the result. □

2.2.3 Triangulated structure on $\mathsf{Ho}\,\mathsf{Comc}\, C$

Reminder on the triangulated structure on $\mathsf{Ho}\,\mathsf{Mod}\, A$

Recall that a Frobenius category is an exact category in the sense of Quillen [Qui73] which has enough injectives and enough projectives and whose class of projectives coincides with that of injectives. It is known [Hel60], [Hap87], [KV87] that the quotient of a Frobenius category $\mathcal{A}$ by the ideal of morphisms factorized by a projective is a triangulated category [Ver77]. It is called the *stable category* associated to $\mathcal{A}$.

Let A be a unital differential graded algebra. The category $\mathsf{Mod}\, A$, endowed with the *class* $\mathcal{E}$ formed of the exact sequences

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

which are split into the category of graded modules, is an exact category. The class of injective objects is made up of complexes of the form

$$IM = \left(M \oplus SM, \begin{bmatrix} 0 & \omega \\ 0 & 0 \end{bmatrix}\right), \quad M \in \mathsf{Mod}\, A.$$

It coincides with the class of projective objects. The category $\mathsf{Mod}\, A$ is therefore a Frobenius category. We denote by $\mathcal{H}A$ the stable category associated with $\mathsf{Mod}\, A$. It is a triangulated category. Its suspension functor is the functor $M \mapsto SM$. Its standard triangles come from the exact sequences of $\mathcal{E}$. The quasi-isomorphisms of $\mathsf{Mod}\, A$ are exactly the morphisms f whose image $\overline{f}$ by the canonical functor $\mathsf{Mod}\, A \to \mathcal{H}A$ fits into a triangle

$$N \to M \xrightarrow{\overline{f}} M' \to SN,$$

where N is acyclic. The *derived category* $\mathcal{D}A$ is the localization of the category $\mathcal{H}A$ with respect to quasi-isomorphisms. The *standard triangles* of $\mathcal{D}A$ are the image under the functor

$$Q : \mathcal{H}A \longrightarrow \mathcal{D}A$$

of standard triangles of $\mathcal{H}A$. The derived category $\mathcal{D}A$, equipped with the suspension endofunctor, is triangulated for the class of *distinguished triangles*, i.e. triangles isomorphic to standard triangles. If f is a morphism of $\mathsf{Mod}\, A$, we denote by $C(f)$ its cone. If

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

is an exact sequence (not necessarily split) of $\mathsf{Mod}\, A$, the morphism $[p, 0] : C(i) \to M''$ is a quasi-isomorphism and the sequence

$$M' \xrightarrow{Q\bar{i}} M \xrightarrow{Q\bar{p}} M'' \xrightarrow{\delta} SM',$$

where the morphism δ is the morphism of $\mathcal{D}A$ defined by

$$M'' \xleftarrow{[p,0]} C(i) \xrightarrow{[-1,0]} SM',$$

is a distinguished triangle of $\mathcal{D}A$.

Triangulated structure on $\mathsf{Ho}\,\mathsf{Comc}\, C$

Let C be an object of Cogca. The category $\mathsf{Comc}\, C$, endowed with the class $\mathcal{F}$ of short exact sequences which are split in the category of graded comodules, is a Frobenius category whose class of injective objects is formed of the objects

$$IN = \left(N \oplus SN, \begin{bmatrix} 0 & \omega \\ 0 & 0 \end{bmatrix}\right), \quad N \in \mathsf{Comc}\, C.$$

We denote by $\mathcal{H}C$ the associated stable category. It is triangulated. Its suspension functor is $N \mapsto SN$. Exact sequences of $\mathcal{F}$ give rise to *standard triangles*. Distinguished triangles are triangles isomorphic to standard triangles.

Let $\tau : C \to A$ be an acyclic admissible twisting cochain where A is an object of Alga. The functors L and R form a pair of exact functors between the categories $\mathsf{Comc}\, C$ and $\mathsf{Mod}\, A$ and preserve injectivity. They therefore induce a pair of triangulated adjoint functors between the stable categories $\mathcal{H}C$ and $\mathcal{H}A$. The *derived category* $\mathcal{D}C$ is the localized category $(\mathcal{H}C)[\mathcal{E}q^{-1}]$. It is clearly isomorphic to the category $\mathsf{Ho}\,\mathsf{Comc}\, C$. Recall (Theorem 2.2.2.2) that the functors R and L (defined in Section 2.2.1) induce inverse equivalences of each other between the localized categories

$$\mathcal{D}A = (\mathcal{H}A)[Qis^{-1}] \quad \text{and} \quad (\mathcal{H}C)[\mathcal{E}q^{-1}] = \mathcal{D}C.$$

In particular, the multiplicative system $\mathcal{E}q$ is compatible with the triangles of $\mathcal{H}C$ because it is the inverse image of the multiplicative system of isomorphisms of $\mathcal{D}A$ by the composite triangulated functor

$$\mathcal{H}C \xrightarrow{L} \mathcal{H}A \longrightarrow \mathcal{D}A.$$

It follows that $\mathcal{D}C$ carries a canonical triangulated structure and that the equivalences induced between $\mathcal{D}A$ and $\mathcal{D}C$ are triangulated functors.

2.2.4 Characterization of the acyclicity of twisting cochains

We recall that the functor $-^+ : \mathsf{Cogc} \to \mathsf{Cogca}$ is an equivalence of categories (Section 2.1.2). Provide Cogca with the model category structure induced by that of Cogc (see Theorem 1.3.1.2).

Proposition 2.2.4.1. Let A be an object of Alga and C an object of Cogca. Let $\tau : C \to A$ be an admissible twisting cochain. The following conditions are equivalent.

a. The twisting cochain τ is acyclic, i.e. if M is an object of $\mathsf{Mod}\, A$, the adjunction morphism

$$\Phi : LRM \to M$$

is a quasi-isomorphism of $\mathsf{Mod}\, A$.

b. If N is an object of $\mathsf{Comc}\, C$, the adjunction morphism

$$\Psi : N \to RLN$$

is a weak equivalence of $\mathsf{Comc}\, C$.

c. The adjunction morphism

$$LRA = A \otimes_\tau C \otimes_\tau A \xrightarrow{\Phi_A} A$$

is a quasi-isomorphism of $\mathsf{Mod}\, A$.

d. The morphism

$$\eta_A \otimes \varepsilon_C : e \to A \otimes_\tau C$$

is a weak equivalence of $\mathsf{Comc}\, C$.

e. The morphism of algebras f_τ (Lemma 2.2.1.5) is a quasi-isomorphism.

f. The morphism de coalgebras g_τ (Lemma 2.2.1.5) is an equivalence of Cogca.

Proof. $a \Rightarrow b$. This is a consequence of point b of theorem 2.2.2.2.

$a \Rightarrow c$. This is clear.

$b \Rightarrow d$. We have the equality $\Psi_e = \eta_A \otimes \varepsilon_C$.

$c \Rightarrow a$. The subcategory of $\mathcal{D}A$ consisting of objects M such that

$$\Phi : LRM \to M$$

is a quasi-isomorphism is a traigulated subcategory with infinite sums containing A by assumption. It thus coincides (see [Kel94a, 4.2]) with $\mathcal{D}A$.

$d \Rightarrow e$. Recall that $\tau_C : C \to \Omega^+ C$ is acyclic (2.2.1.9). This implies that the morphism

$$L_{\tau_C} e = \Omega^+ C \longrightarrow L_{\tau_C}(A \otimes_\tau C) = L_{\tau_C} R_{\tau_C} \mathsf{Res}\, A$$

and the adjunction morphism

$$L_{\tau_C} R_{\tau_C} \mathsf{Res}\, A \to \mathsf{Res}\, A$$

are quasi-isomorphisms. The morphism f_τ is a quasi-isomorphism because it is equal to the composition

$$\Omega^+ C \longrightarrow L_{\tau_C} R_{\tau_C} \mathsf{Res}\, A \longrightarrow \mathsf{Res}\, A.$$

$e \Leftrightarrow f$. This is point b of Theorem 1.3.1.2.

$e \Rightarrow a$. Since the cochain τ_C is acyclic, the adjunction morphism

$$L_{\tau_C} R_{\tau_C} M = M \otimes_\tau C \otimes_{\tau_C} \Omega^+ C \to M$$

is a quasi-isomorphism. Moreover, it is equal to the composition

$$M \otimes_\tau C \otimes_{\tau_C} \Omega^+ C \xrightarrow{\phi_M} M \otimes_\tau C \otimes_\tau A \xrightarrow{\Phi_M} M.$$

Thus, it suffices to show that the morphism ϕ_M induced by the morphism f_τ is a quasi-isomorphism. Endow the comodule $M \otimes_\tau C$ with its primitive filtration. We then have

$$\mathsf{Gr}(M \otimes_\tau C) = M \otimes \mathsf{Gr} C$$

and induced filtrations on $M \otimes_\tau C \otimes_\tau A$ and $M \otimes_\tau C \otimes_\tau \Omega^+ C$ which satisfy

$$\mathsf{Gr}(M \otimes_\tau C \otimes_\tau A) = M \otimes \mathsf{Gr} C \otimes A \quad \text{and} \quad \mathsf{Gr}(M \otimes_\tau C \otimes_{\tau_C} \Omega^+ C) = M \otimes \mathsf{Gr} C \otimes \Omega^+ C.$$

For these filtrations, the morphism ϕ_M is a filtered morphism and it induces quasi-isomorphisms in the graded objects because f_τ is a quasi-isomorphism. Therefore, it is a quasi-isomorphism $\square$

2.3 Polydules

2.3.1 Definitions

Definition 2.3.1.1. Let A be an $\mathrm{A_n}$-algebra. An $\mathrm{A_n}$*-module over* A *in category* $\mathcal{G}r\mathsf{C}'$ is a graded object M in $\mathcal{G}r\mathsf{C}'$ endowed with a family of graded morphisms

$$m_i^M : M \otimes A^{\otimes i-1} \to M, \quad 1 \leq i \leq n,$$

of degree $2-i$, such that an equation $(*'_m)$ of the same form as the equation $(*_m)$ of Definition 1.2.1.1 holds for all $1 \leq m \leq n$. In the equation $(*'_m)$, for $j > 0$, the terms

$$m_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l})$$

of the equation $(*_m)$ must be interpreted as

$$m_i^M(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) : M \otimes A^{\otimes m-1} \to M,$$

and, for $j = 0$, as

$$m_i^M(m_k^M \otimes \mathbf{1}^{\otimes l}) : M \otimes A^{\otimes m-1} \to M.$$

Definition 2.3.1.2. Let A be an A_∞-algebra. An A*-polydule in* $\mathcal{G}r\mathsf{C}'$ (in the literature, this structure is commonly called an A_∞*-module over* A) is a M graded object endowed with a family of graded morphisms

$$m_i^M : M \otimes A^{\otimes i-1} \to M, \quad 1 \leq i,$$

of degree $2-i$, such that the equation $(*'_m)$ holds for all $1 \leq m$.

Definition 2.3.1.3. The *suspension* SM of an A-polydule is the A-polydule whose underlying graded object is the suspension SM and whose multiplications are defined by

$$m_i^{SM} = (-1)^i s \circ m_i^M \circ (\omega \otimes \mathbf{1}^{\otimes i-1}), \quad i \geq 1.$$

The section 2.3.3 will certify that this indeed defines an A-polydule.

Definition 2.3.1.4. Let A be an $\mathrm{A_n}$-algebra, and let M and N be two $\mathrm{A_n}$-modules over A. An $\mathrm{A_n}$*-morphism* of $\mathrm{A_n}$-modules $f : M \to N$ is a family of graded morphisms of C'

$$f_i : M \otimes A^{\otimes i-1} \to N, \quad 1 \leq i \leq n,$$

of degree $1-i$, satisfying, for all $1 \leq m \leq n$, the equality

$$(**'_m) \quad \sum (-1)^{jk+l} f_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = \sum m_{s+1}(f_r \otimes \mathbf{1}^{\otimes s})$$

in $\mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(M \otimes A^{\otimes m-1}, N)$, where $j+k+l = m$, $i = j+1+l$ and $r+s = m$. An $\mathrm{A_n}$-morphism f is *strict* if $f_i = 0$ for all $i \geq 2$. Let M, N and T be three $\mathrm{A_n}$-modules on A. Let $g : M \to N$ and $f : N \to T$ be two $\mathrm{A_n}$-morphisms of $\mathrm{A_n}$-modules. The *composition* $f \circ g : M \to T$ is defined by

$$(f \circ g)_i = \sum_{k+l=i} f_{1+l}(g_k \otimes \mathbf{1}^{\otimes l}), \quad 1 \leq i \leq n.$$

Definition 2.3.1.5. Let A be an A_∞-algebra and let M and N be two A-polydules. An A_∞-*morphism* $f : M \to N$ is a family of graded morphisms

$$f_i : M \otimes A^{\otimes i-1} \to N, \quad 1 \leq i,$$

of degree $1 - i$, such that the equation $(**'_m)$ is satisfied for all $1 \leq m$. The *composition* of A_∞-morphisms is defined by the same formulas as that of the composition of A_n-morphisms. An A_∞-morphism f is *strict* if $f_i = 0$ for all $i \geq 2$.

It will result from Section 2.3.3 that we do indeed obtain a category. We denote it $\mathsf{Nod}_\infty A$. The letter N replaces the letter M of Mod and refers to Non in "Non unital A_∞-module". Let $\mathsf{Nod}_\infty^{\mathsf{strict}} A$ denote the subcategory of $\mathsf{Nod}_\infty A$ whose objects are the A-polydules and whose morphisms are the strict A_∞-morphisms.

Remark 2.3.1.6. Let A be an A_∞-algebra. In a manner analogous to Remark 1.2.1.3, if M is an A-polydule,

- (M, m_1) is a complex;
- the morphism $m_2^M : M \otimes A \to M$ defines an action up to homotopy of the strongly homotopically associative algebra (Remark 1.2.1.3) A on M. The lack of compatibility of the multiplication m_2^A and the action m_2^M is equal to the boundary of m_3^M in

$$(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(M \otimes A^{\otimes 2}, M), \delta),$$

  where δ is defined using m_1^M and m_1^A.
- If $f : M \to N$ is an A_∞-morphism of A-polydules, the morphism f_1 is a morphism of complexes $(M, m_1^M) \to (N, m_1^N)$.

Remark 2.3.1.7. Let A be an A_∞-algebra. The morphisms m_i^A, $i \geq 1$, define an A-polydule structure on the object underlying A.

Remark 2.3.1.8. Let A be an object of Alg and (M, d^M, Δ^M) a differential graded A-module. The morphisms

$$m_1^M = d^M, \quad m_2^M = \Delta^M, \quad m_i^M = 0 \quad \text{for} \quad \mathrm{i} \geq 3$$

define on the object underlying M an A-polydule structure. The category of differential graded A-modules is a nonfull subcategory of the category of A-polydules.

Definition 2.3.1.9. Let A be an A_∞-algebra and let M and N be two A-polydules. An A_∞-morphism of A-polydules $f : M \to N$ is an A_∞-*quasi-isomorphism* if f_1 is a quasi-isomorphism of complexes.

Definition 2.3.1.10. Let A be an A_∞-algebra and let M and N be two A-polydules. Let f and g be two A_∞-morphisms $M \to N$. A *homotopy between* f *and* g is a family of morphisms

$$h_i : M \otimes A^{\otimes i-1} \to N, \qquad 1 \leq i,$$

of degree $-i$ satisfying, for all $1 \leq m$, the equation

$$(***'_m) \qquad f_m - g_m \;=\; \sum (-1)^s m_{1+s}(h_r \otimes \mathbf{1}^{\otimes s}) + \sum (-1)^{jk+l} h_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l})$$

in $\mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(M \otimes A^{\otimes m-1}, N)$, where $r + s = m$ and $j + k + l = m$. Two A_∞-morphisms of A_∞-algebras f and g are *homotopic* if there exists a homotopy between f and g.

2.3.2 Strict units, augmentations and reductions

In this chapter, we will study *strictly unital* polydules over *augmented* A_∞-algebras. We will therefore define here a type of unitality for A_∞-structures: *strict unitality*. This structure will allow us to generalize certain properties of unital modules to polydules. The relevance of this notion of unitality relative to the homotopy of A_∞-structures will be the subject of Chapter 3.

Definition 2.3.2.1. An A_∞-algebra A is *strictly unital* if it is endowed with a graded morphism $\eta : e \to A$ of degree 0 such that $m_i(\mathbf{1} \ldots \mathbf{1} \otimes \eta \otimes \mathbf{1} \ldots \mathbf{1}) = 0$ for all $i \neq 2$ and

$$m_2(\mathbf{1}_A \otimes \eta) = m_2(\eta \otimes \mathbf{1}_A) = \mathbf{1}_A.$$

The morphism η is called the (strict) *unit* of A. If A and A' are two strictly unital A_∞-algebras, an A_∞-morphism $f : A \to A'$ is *strictly unital* if $f_1\eta^A = \eta^{A'}$ and $f_i(\mathbf{1} \ldots \mathbf{1} \otimes \eta \otimes \mathbf{1} \ldots \mathbf{1}) = 0$ for all $i \geq 2$.

By the remark 1.2.1.5, a unital differential graded algebra is a strictly unital A_∞-algebra. In particular, the algebra e is a strictly unital A_∞-algebra.

Definition 2.3.2.2. An A_∞-algebra A is *augmented* if it is strictly unital and endowed with a strict A_∞-morphism of strictly unital A_∞-algebras $\varepsilon : A \to e$. The morphism ε is called the *augmentation* of A.

The *reduced* A_∞*-algebra* $\overline{A}$ is the kernel of ε. Let A be an A_∞-algebra. The *augmented* A_∞*-algebra* A^+ has the underlying object $A \oplus e$, its multiplications m_i, $i \geq 1$, are such that the canonical injection $e \to A \oplus e$ is the strict unit and such that they coincide with m_i^A, $i \geq 1$, on A. Its augmentation is the canonical projection $A \oplus e \to e$. We denote by Alga_∞ the *category* of augmented A_∞-algebras. The *augmentation functor* $\mathsf{Alg}_\infty \to \mathsf{Alga}_\infty$ is an equivalence whose quasi-inverse is the *reduction functor*.

Definition 2.3.2.3. Let A be a strictly unital A_∞-algebra. An A-polydule M is *strictly unital* if $m_i^M(\mathbf{1}_M \otimes \mathbf{1} \ldots \mathbf{1} \otimes \eta \otimes \mathbf{1} \ldots \mathbf{1}) = 0$ for all $i \geq 3$ and

$$m_2^M(\mathbf{1}_M \otimes \eta) = \mathbf{1}_M.$$

A *strictly unital* morphism between strictly unital A-polydules is an A_∞-morphism f of A-polydules such that

$$f_i(\mathbf{1}_M \otimes \mathbf{1} \ldots \mathbf{1} \otimes \eta \otimes \mathbf{1} \ldots \mathbf{1}) = 0, \quad i \geq 2.$$

If f and g are two strictly unital morphisms, a homotopy h between f and g is *strictly unital* if

$$h_i(\mathbf{1}_M \otimes \mathbf{1} \ldots \mathbf{1} \otimes \eta \otimes \mathbf{1} \ldots \mathbf{1}) = 0, \quad i \geq 2.$$

If h is a strictly unital homotopy between two strictly unital morphisms f and g, we say that f and g are *homotopic* (relative to h) and we denote $f \sim g$. We denote by $\mathsf{Mod}_\infty A$ the *category* of strictly unital A-polydules whose morphisms are strictly unital morphisms and by $\mathsf{Mod}_\infty^{\mathsf{strict}} A$ the *category* of strictly unital A-polydules whose morphisms are strict and strictly unital A_∞-morphisms.

If A is an A_∞-algebra and M an A-polydule, M^+ is the (strictly unital) A^+*-polydule* whose object is underlying M and whose multiplication $m_i^{M^+}$, $i \geq 1$, is such that, restricted to A, it coincides with m_i^M, $i \geq 1$ (in particular the m_1 does not change). This defines an isomorphism

$$^+ : \mathsf{Nod}_\infty A \xrightarrow{\sim} \mathsf{Mod}_\infty A^+$$

compatible with homotopy. The quasi-inverse is given by the functor which sends M to the $\overline{A}$-polydule $\overline{M}$ whose underlying object is M and whose multiplication $m_i^{\overline{M}}$, $i \geq 2$, is the restriction of m_i^M, $i \geq 2$, to $M \otimes A^{\otimes i-1}$.

2.3.3 Bar construction

The proofs of this section being almost identical to those of the section 1.2.2, we content ourselves with stating the results.

Bar construction of polydules

Let A and M be two graded objects. For each $i \geq 1$, we define a bijection

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(M \otimes A^{\otimes i-1}, M) & \to & \mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(SM \otimes (SA)^{\otimes i-1}, SM) \\ m_i^M & \mapsto & b_i^M \end{array}$$

by the relation

$$\omega \circ b_i^M = -m_i^M \circ \omega^{\otimes i} \qquad \text{(where} \quad \omega = s^{-1}\text{)}.$$

Let A be an A_∞-algebra. We recall (2.1.2.1) that a differential b^M on the (co-unital) graded $(BA)^+$-comodule $SM \otimes (BA)^+$ is determined by the composition

$$(\mathbf{1} \otimes \eta^{(BA)^+}) \circ b^M : SM \otimes (BA)^+ \to SM$$

whose components we denote b_i^M, $i \geq 1$. The bijections $m_i^M \leftrightarrow b_i^M$ induce a bijection from the set of structures of A-polydule on M to the set of differentials b^M on the graded $(BA)^+$-comodule $SM \otimes (BA)^+$.

Let A, M and N be three graded objects. For each $i \geq 1$, we define a bijection

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(M \otimes A^{\otimes i-1}, M) & \to & \mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(SM \otimes (SA)^{\otimes i-1}, SM) \\ f_i & \to & F_i \end{array}$$

by the relations

$$\omega \circ F_i = (-1)^{|F_i|} f_i \circ \omega^{\otimes i}, \quad i \geq 1,$$

where F_i is a graded morphism of degree $|F_i|$. Let A be an A_∞-algebra. We recall (2.1.2.1) that a graded morphism of (co-unital) $(BA)^+$-comodules

$$F : SM \otimes (BA)^+ \to SN \otimes (BA)^+$$

is determined by the composition

$$(\mathbf{1} \otimes \eta^{(BA)^+}) \circ F : SM \otimes (BA)^+ \to SM$$

whose components we denote F_i, $i \geq 1$. The bijections $f_i \leftrightarrow F_i$ induce a bijection of the product of sets of graded morphisms

$$f_i : M \otimes A^{\otimes i-1} \to N, \quad i \geq 1,$$

of degree $1 - i + n$, on the set of graded morphisms of $(BA)^+$-comodules $F : SM \otimes (BA)^+ \to SN \otimes (BA)^+$ of degree n. If M and N are A-polydules, this bijection sends bijectively the set of families defining an A_∞-morphism $f : M \to N$ to the set of differential graded morphisms of $(BA)^+$-comodules

$$F : SM \otimes (BA)^+ \to SN \otimes (BA)^+.$$

If f and g are two A_∞-morphisms of A-polydules, the same bijection sends bijectively the set of homotopies between f and g to the set of homotopies between the morphisms of $(BA)^+$-comodules F and G corresponding to f and g.

This gives us a functor

$$\mathsf{Nod}_\infty A \to \mathsf{Comc}(BA)^+, \quad M \mapsto (SM \otimes (BA)^+, b^M).$$

Bar construction of strictly unital polydules over an augmented A_∞-algebra

Let A be an augmented A_∞-algebra. We denote by B^+A the co-augmented coalgebra $(B\overline{A})^+$, where $\overline{A}$ is the reduced A_∞-algebra associated to A. Be careful not to confuse the co-augmented coalgebras B^+A and $(BA)^+$.

By the section 2.3.2, the functor $N \mapsto \overline{N}$ is an isomorphism of categories

$$\mathsf{Mod}_\infty A \xrightarrow{\sim} \mathsf{Nod}_\infty \overline{A}.$$

The composite functor

$$B_A : \mathsf{Mod}_\infty A \xrightarrow{\sim} \mathsf{Nod}_\infty \overline{A} \to \mathsf{Comc}\, B^+A$$

is called the *bar construction* functor. The suspension SM of a polydule is sent by the bar construction to $BSN = (S^2N \otimes B^+A, b^{SN})$. We check that the latter is isomorphic to SBN. The bar construction functor sends homotopic A_∞-morphisms to homotopic morphisms of comodules and it induces an equivalence between the category $\mathsf{Mod}_\infty A$ and the subcategory $\mathsf{prcol}\, B^+A$ of $\mathsf{Comc}\, B^+A$ made up of almost free objects.

2.3.4 Enveloping algebra

In this section, we define the enveloping algebra UA of an augmented A_∞-algebra A and then show that the category $\mathsf{Mod}\, UA$ is isomorphic to the category $\mathsf{Mod}_\infty^{\mathsf{strict}} A$.

Let V be a graded (resp. differential graded) vector space. The *(augmented) tensor algebra* TV is the augmentation $(\overline{T}V)^+$ of the reduced tensor algebra. Let $i : V \to TV$ be the canonical injection.

Lemma 2.3.4.1. Let M be a graded object. The map $\mu^M \mapsto \mu^M(\mathbf{1} \otimes i)$ is a bijection from the set of unital TV-module structures on M to the set of graded morphisms

$$M \otimes V \to M$$

of degree 0. The inverse map associates to g the multiplication

$$\mu : M \otimes TV \to M$$

whose component $M \otimes e \to M$ is the identity and component $M \otimes V^{\otimes i} \to M$ is the morphism $g \circ (g \otimes \mathbf{1}) \circ \cdots \circ (g \otimes \mathbf{1}^{\otimes i-1})$.

Definition 2.3.4.2. Let A be an augmented A_∞-algebra. The *enveloping algebra* of A is the differential graded algebra $UA = \Omega^+ B^+ A$, that is, the algebra $(\Omega B\overline{A})^+$.

Lemma 2.3.4.3. The A_∞-morphism $A \to UA$ given by the adjunction morphism

$$B^+A \to B^+UA = B^+\Omega^+B^+A$$

is an A_∞-quasi-isomorphism. It is universal among the A_∞-morphisms from A to a differential graded algebra. □

Proof. It is an A_∞-quasi-isomorphism by Lemma 1.3.3.6. The universality is immediate thanks to the adjunction (Ω, B). □

Lemma 2.3.4.4. We have an isomorphism of categories

$$i : \mathsf{Mod}\, UA \to \mathsf{Mod}_\infty^{\mathsf{strict}}\, A, \quad M \to S^{-1}M.$$

Proof. Let M be a graded object. We are going to show that the unital UA-module structures over SM are the strictly unital A-polydule structures over M. Let m_1^M be a differential over M and let

$$m_i^M : M \otimes A^{\otimes i-1} \to M, \quad i \geq 2,$$

graded morphisms of degree $2 - i$. We define using the bijections $m_i^M \leftrightarrow b_i^M$ of section 1.2.2, a morphism

$$g : SM \otimes (B\overline{A}) \to SM.$$

By lemma 2.3.4.1, the morphism

$$SM \otimes S^{-1}(B\overline{A}) \xrightarrow{\mathbf{1}\otimes s} SM \otimes (B\overline{A}) \xrightarrow{g} SM$$

lifts to a graded unital Ω^+B^+A-module structure μ^U on SM. We verify that (SM, μ^U, Sm_1) defines a unital differential graded module if and only if the m_i^M, $i \geq 1$, define a strictly unital A-polydule structure on M. If SM and SN are two UA-modules, the morphisms of UA-modules $SM \to SN$ are clearly identified with the strict A_∞-morphisms of A-polydules $M \to N$. □

2.4 Derived category of an augmented A_∞-algebra

Introduction

Let A be an augmented A_∞-algebra. The purpose of this section is to show that the derived category

$$\mathcal{D}_\infty A = \mathsf{Mod}_\infty A[Qis^{-1}]$$

is equivalent to the categories

$$\mathcal{H}_\infty A = \mathsf{Mod}_\infty A/\!\sim \quad \text{and} \quad \big(\mathsf{Mod}_\infty^{\mathsf{strict}} A\big)[Qis^{-1}]$$

where $\sim$ is the homotopy relation. The derived category of any A_∞-algebra is studied in chapter 4.

Section plan

This section is divided into three subsections. In subsection 2.4.1, we prove the homotopy theorem and the A_∞-quasi-isomorphism theorem for polydules. For this, we will characterize the fibrant objects of the model category $\mathsf{Comc}\, B^+A$: *they are exactly the direct factors of almost cofree objects* and we show that the above theorems then appear as particular cases of fundamental results of Quillen's homotopic algebra (see appendix A). In the subsection 2.4.2, we show the equivalences announced in the introduction above (again thanks to Quillen's homotopic algebra). In section 2.4.3, we study the triangulated structure of $\mathcal{D}_\infty A$.

2.4.1 Fibrant objects of $\mathsf{Comc}\, B^+A$

Let A be an object of Alga_∞. The purpose of this section is to show the following proposition:

Proposition 2.4.1.1.

a. The homotopy relation (2.3.2.3) in $\mathsf{Mod}_\infty A$ is an equivalence relation compatible with composition.

b. An A_∞-quasi-isomorphism of A-polydules is a homotopy equivalence.

c. Let A' be an object of Alga. Let $\mathsf{Modsh}\, A'$ be the full subcategory of $\mathsf{Mod}_\infty A'$ consisting of unital differential graded A'-modules. Let $\sim$ denote the homotopy relation on $\mathsf{Modsh}\, A'$. The inclusion $\mathsf{Mod}\, A' \hookrightarrow \mathsf{Modsh}\, A'$ induces an equivalence

$$\mathcal{D}A' \xrightarrow{\sim} \mathsf{Modsh}\, A'/\!\sim .$$

Remark 2.4.1.2. Part *c* remains true even in the case where the unital differential graded algebra A' is not augmented (see 4.1.3.8)

Proof. The proof is identical to that of corollary 1.3.1.3. It proceeds in the same way by using (instead of the main theorem 1.3.1.2) the theorem 2.2.2.2 and the proposition 2.4.1.3 below. $\square$

A refinement of the characterization of fibrant objects of Theorem 2.2.2.2

Let C be an object of Cogca. Equip the $\mathsf{Comc}\, C$ category with its canonical model category structure (2.2.2.4). Let $\tau : C \to A'$ be an admissible acyclic twisting cochain, where A' is an object of Alga (there always exists such a cochain thanks to Lemma 2.2.1.9). Theorem 2.2.2.2 says that the fibrant objects of $\mathsf{Comc}\, C$ are the direct factors of objects of the form $R_\tau M$, where M is an object of $\mathsf{Mod}\, A'$. In particular, the fibrant objects are direct factors of almost cofree objects of $\mathsf{Comc}\, C$. Let us show that the converse is true for some coalgebras:

Proposition 2.4.1.3. Let C be an object of Cogca which is isomorphic, as a graded coalgebra, to a tensor coalgebra. The fibrant objects of $\mathsf{Comc}\, C$ are exactly the direct factors of almost cofree objects.

In particular, since the coalgebra C is isomorphic to the bar construction B^+A of an object A of Alga_∞ the fibrant objects of $\mathsf{Comc}\, C$ are exactly the direct factors of comodules which are in the image of the bar construction of an A-polydule. The proof of this result is postponed to the end of this section. We first demonstrate some propositions.

$\mathsf{Mod}_\infty A$ as a "model category without limits"

Let A be an augmented A_∞-algebra. In the category $\mathsf{Mod}_\infty A$, we consider the following three classes of morphisms:

- the class $\mathcal{E}q$ of *weak equivalences*, i.e., A_∞-quasi-isomorphisms,
- the class $\mathcal{C}of$ of *cofibrations*, i.e., A_∞-morphisms $j : M \to M'$ such that j_1 is a monomorphism,
- the class $\mathcal{F}ib$ of *fibrations*, i.e., A_∞-morphisms $q : M \to M'$ such that q_1 is an epimorphism.

Theorem 2.4.1.4. The category $\mathsf{Mod}_\infty A$, equipped with the three classes defined above, satisfies axiom (A) of Theorem 1.3.3.1 and axioms (CM2) – (CM5) of Definition A.7. All objects are fibrant and cofibrant.

Proof. This is identical to that of 1.3.3.1 because it is based on the obstruction lemmas (see appendix B.2). $\square$

Links between the "model category without limits" $\mathsf{Mod}_\infty A$ and the model category $\mathsf{Comc}\, B^+A$

Proposition 2.4.1.5. Let M and M' be two objects in $\mathsf{Mod}_\infty A$.

a. An A_∞-morphism $f : M \to M'$ is an A_∞-quasi-isomorphism in $\mathsf{Mod}_\infty A$ if and only if the morphism $Bf : BM \to BM'$ is a weak equivalence in $\mathsf{Comc}\, B^+A$.

b. An A_∞-morphism $j : M \to M'$ is a cofibration in $\mathsf{Mod}_\infty A$ if and only if $Bj : BM \to BM'$ is a cofibration in $\mathsf{Comc}\, B^+A$.

c. An A_∞-morphism $q : M \to M'$ is a fibration in $\mathsf{Mod}_\infty A$ if and only if $Bq : BM \to BM'$ is a fibration in $\mathsf{Comc}\, B^+A$.

Proof. Let UA be the enveloping algebra of A. Recall (2.2.1.5) that the universal twisting cochain

$$\tau : B^+A \to \Omega^+B^+A = UA$$

is acyclic. By Corollary 2.2.2.3, we have a Quillen equivalence

$$(L, R) : \mathsf{Comc}\, B^+A \to \mathsf{Mod}\, UA.$$

a. If f is an A$_\infty$-quasi-isomorphism, the morphism Bf is a filtered quasi-isomorphism for primitive filtrations. By Lemma 2.2.2.5, it is a weak equivalence in $\mathsf{Comc}\, B^+A$. Suppose that Bf is a weak equivalence of $\mathsf{Comc}\, B^+A$. Consider the diagram of $\mathsf{Comc}\, B^+A$

$$\begin{tikzcd} BM \arrow[r] \arrow[d, "Bf"'] & RLBM \arrow[d, "RLBf"] \\ BM' \arrow[r] & RLBM'. \end{tikzcd}$$

As $R = Bi$, this diagram is the image under B of a diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & iLBM \arrow[d, "iLBf"] \\ M' \arrow[r] & iLBM'. \end{tikzcd}$$

As Bf is a weak equivalence of $\mathsf{Comc}\, B^+A$, the morphism LBf is a quasi-isomorphism of $\mathsf{Mod}\, UA$. The (strict) morphism $iLBf$ is thus an A$_\infty$-quasi-isomorphism in $\mathsf{Mod}_\infty A$. The lemma 2.4.1.6 below shows that the horizontal arrows of the diagram above represent A$_\infty$-quasi-isomorphisms. By the saturation property of A$_\infty$-quasi-isomorphisms in $\mathsf{Mod}_\infty A$, f is therefore a A$_\infty$-quasi-isomorphism.

b and *c*. Same proof as in proposition 1.3.3.5. □

Lemma 2.4.1.6. Let M be an object of $\mathsf{Mod}_\infty A$. The adjunction morphism $BM \to RLBM$ induces a quasi-isomorphism in the primitives.

Proof. We need to show that the morphism

$$SM \to SM \otimes B^+A \otimes UA$$

is a quasi-isomorphism. Let C be the coalgebra B^+A. Recall that by definition $\Omega^+C = \Omega\overline{C}$. It remains to show that

$$SM \to SM \otimes C \otimes \Omega^+C$$

is a quasi-isomorphism. Let us endow Ω^+C with the filtration induced by the primitive filtration of $\overline{C}$ considered as coalgebra. We then have a filtration of Ω^+C defined by the sequence

$$\left(\Omega^+C\right)_i = (\Omega\overline{C})_i \oplus e, \quad i \geq 0.$$

Endow C, considered as an object of $\mathsf{Com}\, C$, with its primitive filtration as a C-module (completing it with $C_{[0]} = e$). We also endow M with the filtration defined by the sequence $M_i = M$, $i \geq 0$. These filtrations induce a filtration of complexes on $SM \otimes C \otimes \Omega^+C$. Just as at the end of the proof of point b of Lemma 2.2.1.9, we show that

$$\mathsf{Gr}_0(SM \otimes C \otimes \Omega^+C) = SM, \quad \mathsf{Gr}_i(SM \otimes C \otimes \Omega^+C) = 0 \quad \text{for} \quad i \geq 1.$$

□

Proof of Proposition 2.4.1.3. We can assume that C is equal to B^+A, for A an augmented A_∞-algebra. Let τ be the universal twisting cochain of B^+A. We know that the fibrant objects of $\mathsf{Comc}\, B^+A$ are direct summands[1] of objects of the form $RM = M \otimes_\tau B^+A$, where M is an object of $\mathsf{Mod}\,\Omega^+B^+A$. Therefore, they are direct summands[2] of almost cofree objects. Conversely, if N is an almost cofree object, it is isomorphic to the image under the bar construction of an object M in $\mathsf{Mod}_\infty A$. Since this latter object is fibrant in $\mathsf{Mod}_\infty A$, the object N is fibrant in $\mathsf{Comc}\, B^+A$ by point c of Proposition 2.4.1.5. □

2.4.2 The derived category $\mathcal{D}_\infty A$

In this section, we define the derived category $\mathcal{D}_\infty A$ and give several descriptions of it.

The point $a.$ of Proposition 2.4.1.1 shows that the following definition makes sense.

Definition 2.4.2.1. Let A be an augmented A_∞-algebra. We denote by $\mathcal{H}_\infty A$ the category $\mathsf{Mod}_\infty A/\sim$, where $\sim$ is the homotopy relation (see 2.3.2.3). The *derived category* $\mathcal{D}_\infty A$ *of* $\mathsf{Mod}_\infty A$ is the localization of the category $\mathsf{Mod}_\infty A$ with respect to the A_∞-quasi-isomorphisms.

Proposition (2.4.1.1) leads to the following result:

Corollary 2.4.2.2. The canonical projection

$$\mathcal{H}_\infty A \to \mathcal{D}_\infty A$$

is an isomorphism.

Proof. The A_∞-quasi-isomorphisms being homotopy equivalences, the canonical projection

$$\mathcal{H}_\infty A \to \big(\mathcal{H}_\infty A\big)[\mathcal{E}q^{-1}] \simeq \mathcal{D}_\infty A$$

is an equivalence. □

Lemma 2.4.2.3. The composition of functors (see 2.3.4.4)

$$J : \mathsf{Mod}\, UA \xrightarrow{i} \mathsf{Mod}_\infty^{\mathsf{strict}} A \hookrightarrow \mathsf{Mod}_\infty A$$

induces an isomorphism $\mathcal{D}UA \to \mathcal{D}_\infty A$.

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathsf{Mod}\, UA \arrow[r, "J"] \arrow[d, "R"'] & \mathsf{Mod}_\infty^{\mathsf{strict}} A \arrow[d, "\text{incl.}"] \\ \mathsf{Comc}\, B^+A & \mathsf{Mod}_\infty A \arrow[l, "B"] \end{tikzcd}$$

and the functors J, R and B induce equivalences between the categories

$$\mathcal{D}UA, \quad \mathcal{D}C, \quad (\mathsf{Mod}_\infty^{\mathsf{strict}} A)[\mathcal{E}q^{-1}] \quad \text{and} \quad \mathcal{D}_\infty A.$$

□

[1] "facteurs"
[2] ditto

2.4.3 Triangulated structure on $\mathcal{D}_\infty A$

Exact sequences of $\mathsf{Mod}_\infty A$

The functor

$$i : \mathsf{Mod}\, UA \to \mathsf{Mod}_\infty A, \quad SM \mapsto M,$$

identifies (see 2.3.4.4) the category $\mathsf{Mod}\, UA$ with the subcategory $\mathsf{Mod}_\infty^{\mathsf{strict}} A$ of $\mathsf{Mod}_\infty A$. It sends the suspension of an UA-module to the suspension of an A-polydule (see 2.3.1.3). It identifies the short exact sequences of $\mathsf{Mod}\, UA$ which are split in the category of graded modules to the sequences of $\mathsf{Mod}_\infty A$ formed from strict A_∞-morphisms

$$(*) \qquad\qquad M' \xrightarrow{j} M \xrightarrow{q} M'',$$

such that

$$0 \to M' \xrightarrow{j_1} M \xrightarrow{q_1} M'' \to 0$$

is an exact sequence in $\mathcal{C}\mathsf{C}'$ and such that there exists a retraction ρ of j_1 in $\mathcal{G}r\mathsf{C}'$ such that, for all $i \geq 2$,

$$\rho m_i^M = m_i^{M'}(\rho \otimes \mathbf{1}^{\otimes i-1}).$$

Triangulated structure on $\mathcal{D}_\infty A$

We endow the derived category $\mathcal{D}_\infty A$ with the unique triangulated structure (unique up to triangulated equivalence) for which the equivalence

$$J : \mathcal{D}UA \to \mathcal{D}_\infty A$$

of Lemma 2.4.2.3 is triangulated. As the functors

$$R : \mathcal{D}UA \to \mathcal{D}B^+A \quad \text{and} \quad B : \mathcal{D}_\infty A \to \mathcal{D}B^+A$$

are triangulated functors, we deduce the following theorem.

Theorem 2.4.3.1. The triangulated structure on $\mathcal{D}_\infty A$ has as its suspension endofunctor the one defined in 2.3.1.3. The distinguished triangles are precisely those that are isomorphic to the triangles arising from exact sequences of the form $(*)$ in $\mathsf{Mod}_\infty A$. □

Cone of an A_∞-morphism.

If $f : M \to M'$ is an A_∞-morphism of A-polydules, its *cone* $C(f)$ is the A-polydule $M' \oplus SM$ whose multiplications

$$m_i^{C(f)} : (M' \oplus SM) \otimes A^{\otimes i-1} \to M' \oplus SM, \quad i \geq 1,$$

are given by the morphisms

$$m_i^{M'}, \quad m_i^{SM} \text{ (see 2.3.1.3)} \quad \text{and} \quad f_i \circ (\omega \otimes \mathbf{1}^{\otimes i-1}).$$

The bar construction sends $C(f)$ to the cone of Bf.

Lemma 2.4.3.2. Let A be an object of Alga. The inclusion

$$\mathsf{Mod}\, A \hookrightarrow \mathsf{Mod}_\infty\, A$$

induces an triangulated equivalence

$$\mathcal{D}A \to \mathcal{D}_\infty A.$$

Proof. Since $A \to UA$ (see 1.3.3.6) is a quasi-isomorphism, we have a triangulated equivalence between the category $\mathcal{D}A$ and the category $\mathcal{D}UA$. The inclusion (2.3.4.4)

$$i : \mathsf{Mod}\, UA \to \mathsf{Mod}_\infty\, A$$

induces a triangulated equivalence from $\mathcal{D}UA$ to $\mathcal{D}_\infty A$, from which we deduce the result. $\square$

2.5 Derived category of bipolydules (the augmented case)

Introduction

Let A and A'' be two augmented A_∞-algebras. In this section, we define the derived category $\mathcal{D}_\infty(A, A'')$ of strictly unital A-A''-bipolydules and we give several descriptions of it. The case where A and A'' are arbitrary will be treated in chapter 4.

Notations

Let $(\mathsf{C}, \otimes, e)$ and $(\mathsf{C}'', \otimes, e)$ be two semisimple monoidal Grothendieck $\mathbb{K}$-categories and C' a semisimple Grothendieck $\mathbb{K}$-category (not necessarily monoidal). We suppose that C is braided (see [ML98, Chap. XI]). We denote by $\otimes^{op}$ the tensor product of C defined by

$$A \otimes^{op} B = B \otimes A.$$

Suppose that the monoidal category C acts on the left on C' and the monoidal category C'' acts on the right on C' in a compatible manner i.e. C' is equipped with two functors ($\mathbb{K}$-bilinear on the morphism spaces)

$$\begin{array}{rcl} \mathsf{C}' \times \mathsf{C}'' & \to & \mathsf{C}', \\ (M', A'') & \mapsto & M' \otimes A'' \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathsf{C} \times \mathsf{C}' & \to & \mathsf{C}', \\ (A, M') & \mapsto & A \otimes M' \end{array}$$

associative and unital up to given isomorphisms (see [ML98, Chap. XI]) and such that

$$(A \otimes M') \otimes A'' = A \otimes (M' \otimes A'').$$

We further suppose that that we have a semisimple monoidal Grothendieck $\mathbb{K}$-category $\mathsf{C} \otimes \mathsf{C}''$, equipped with a monoidal functor

$$(\mathsf{C}, \otimes^{op}) \times (\mathsf{C}'', \otimes) \to \mathsf{C} \otimes \mathsf{C}'', \quad (A, A'') \mapsto A \otimes A'',$$

$$\mathsf{Hom}_{\mathsf{C}}(A, B) \times \mathsf{Hom}_{\mathsf{C}''}(A'', B'') \to \mathsf{Hom}_{\mathsf{C} \otimes \mathsf{C}''}((A \otimes A''), (B \otimes B'')),$$

bilinear on morphism spaces, with an action on C' and with an isomorphism

$$M \otimes (A \otimes A'') = A \otimes M \otimes A''.$$

The following example appears naturally in the study of A_∞-categories (5.1.1).

Example 2.5.0.1. Let $\mathbb{A}$ and $\mathbb{B}$ be two sets considered as discrete categories. We denote by $\mathsf{C}(\mathbb{A}, \mathbb{B})$ the category of functors

$$\mathbb{B}^{op} \times \mathbb{A} \to \mathsf{Vect}\mathbb{K}.$$

Set

$$\mathsf{C} = \mathsf{C}(\mathbb{A}, \mathbb{A}), \quad \mathsf{C}' = \mathsf{C}(\mathbb{A}, \mathbb{B}) \quad \text{and} \quad \mathsf{C}'' = \mathsf{C}(\mathbb{B}, \mathbb{B}).$$

The tensor products over $\mathbb{A}$ and the $\mathbb{B}$ define the monoidal (braided) category structures on C and C'' and the actions of C and C'' on C'. The category $\mathsf{C} \otimes \mathsf{C}''$ is the category $\mathsf{C}(\mathbb{A} \times \mathbb{B}, \mathbb{A} \times \mathbb{B})$ of functors

$$\big(\mathbb{A} \times \mathbb{B}\big)^{op} \times \big(\mathbb{A} \times \mathbb{B}\big) \to \mathsf{Vect}\mathbb{K}.$$

The functor

$$\mathsf{C}(\mathbb{A}, \mathbb{A}) \times \mathsf{C}(\mathbb{B}, \mathbb{B}) \to \mathsf{C}(\mathbb{A} \times \mathbb{B}, \mathbb{A} \times \mathbb{B})$$

sends (L, M) to the functor

$$(A, B, A', B') \mapsto L(A, A') \otimes_{\mathbb{K}} M(B, B').$$

2.5.1 Definitions of bipolydules

Let A and A'' be two A_∞-algebras of C and C''.

Definition 2.5.1.1. An A_n-$\mathrm{A}_{n'}$-*bimodule on A and A''* is an object of $\mathcal{G}r\mathsf{C}'$ equipped with a family of graded morphisms in $\mathcal{G}r\mathsf{C}'$

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A''^{\otimes j} \to M, \quad 0 \leq i \leq n, \quad 0 \leq j \leq n',$$

of degree $1 - i - j$, such that an equation $(*''_{r,t})$ of the same form as the equation $(*_{r+1+t})$, $r + 1 + t \geq 1$, of the definition 1.2.1.1 holds for all $0 \leq r \leq n$ and $0 \leq t \leq n'$. If M and M' are two A_n-$\mathrm{A}_{n'}$-bimodules on A and A'', a *morphism*

$$f : M \to M'$$

is a family of graded morphisms in $\mathcal{G}r\mathsf{C}'$

$$f_{i,j} : A^{\otimes i} \otimes M \otimes A''^{\otimes j} \to M', \quad 0 \leq i \leq n, \quad 0 \leq j \leq n',$$

of degree $-i - j$, satisfying the equalities $(**'')_{r,t}$, $0 \leq r \leq n$ and $0 \leq t \leq n'$, the morphisms[3]

$$A^{\otimes r} \otimes M \otimes A''^{\odot t} \to M', \quad 0 \leq r \leq n, \quad 0 \leq t \leq n',$$

$$\begin{aligned}\sum(-1)^{\alpha(-i-j)} m_{\alpha,\beta}(\mathbf{1}^{\otimes\alpha} \otimes f_{k,l} \otimes \mathbf{1}^{\otimes\beta}) = \\ \sum(-1)^{j+i(|m_\bullet|)} f_{\bullet,\bullet}(\mathbf{1}^{\otimes i} \otimes m_\bullet \otimes \mathbf{1}^{\otimes j})\end{aligned}$$

where $|m_\bullet|$ is the degree of $m_\bullet$; the $m_\bullet$ must be properly interpreted as $m_\bullet^A$, $m_\bullet^{A''}$ or $m_{\bullet,\bullet}$ according to its place. The composition $g \circ f$ of two morphisms f and g is defined as follows

$$(g \circ f)_n = \sum(-1)^{\alpha(-i-j)} g_{i,j}(\mathbf{1}^{\otimes\alpha} \otimes f_{k,l} \otimes \mathbf{1}^{\otimes\beta}), \quad n \geq 1.$$

[3]Below, Is the power $\odot t$ a typo? Maybe the power should be $\otimes t$?

Definition 2.5.1.2. An A-A''-*bipolydule in* C' (commonly called an A_∞-*bimodule over* A *and* A'' in the literature) is an object of $\mathcal{G}r\mathsf{C}'$ equipped with a family of graded morphisms in $\mathcal{G}r\mathsf{C}'$

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A''^{\otimes j} \to M, \quad i, j \geq 0,$$

of degree $1 - i - j$, such that the equation $(*''_{n,n'})$, is satisfied for $n, n' \geq 0$. If M and M' are two A-A''-polydules, a *morphism*

$$f : M \to M'$$

is a family of graded morphisms in $\mathcal{G}r\mathsf{C}'$ such that the equality $(**'')_{n,n'}$, is satisfied for $n+1+n' \geq 1$. The composition $g \circ f$ of two A_∞-morphisms f and g is defined by the same formulas as in the case of the morphisms of A_n-$\mathrm{A}_{n'}$-bimodules on A and A''. We thus obtain a *category* $\mathsf{Nod}_\infty(A, A'')$. The letter N in Nod_∞ replaces the letter M in Mod_∞ and refers to the N in "Not (necessarily) unital A_∞-bimodules".

We now assume that A and A'' are augmented.

Definition 2.5.1.3. An A-A''-bipolydule is *strictly unital* if for all $i, j \geq 0$, we have

$$m_{i,j}(\mathbf{1}^{\otimes \alpha} \otimes \eta \otimes \mathbf{1}^{\otimes \beta}) = 0, \quad \alpha \neq i, \quad (i,j) \notin \{(0,1),(1,0)\}$$

and

$$m_{1,0} \circ (\eta \otimes \mathbf{1}) = m_{0,1} \circ (\mathbf{1} \otimes \eta) = \mathbf{1}.$$

We denote by $\mathsf{Mod}_\infty(A, A'')$ the category of strictly unital A-A''-bipolydules. It is isomorphic to the category of $\overline{A}$-$\overline{A''}$-bipolydules, where $\overline{A}$ and $\overline{A''}$ are the reductions of A and A''.

Bar construction

We define the bijections

$$\begin{array}{rcl} \mathsf{Hom}((S\overline{A})^{\otimes i} \otimes SM \otimes (S\overline{A''})^{\otimes j}, SM) & \xrightarrow{\sim} & \mathsf{Hom}(\overline{A}^{\otimes i} \otimes M \otimes \overline{A''}^{\otimes j}, M), \\ m_{i,j} & \mapsto & b_{i,j} \end{array}$$

$$\begin{array}{rcl} \mathsf{Hom}((S\overline{A})^{\otimes i} \otimes SM \otimes (S\overline{A''})^{\otimes j}, SM) & \xrightarrow{\sim} & \mathsf{Hom}(\overline{A}^{\otimes i} \otimes M \otimes \overline{A''}^{\otimes j}, M), \\ f_{i,j} & \mapsto & F_{i,j} \end{array}$$

by the relations

$$\omega \circ b_{i,j} = -m_{i,j} \circ \omega^{\otimes i+1+j} \quad \text{and} \quad \omega \circ F_{i,j} = (-1)^{|F_{i,j}|} f_{i,j} \circ \omega^{\otimes i+1+j}.$$

These bijections define the fully faithful *bar construction* functor

$$B : \mathsf{Mod}_\infty(A, A'') \longrightarrow \mathsf{Comc}(B^+A, B^+A''),$$

where $\mathsf{Comc}(B^+A, B^+A'')$ is the *category* of objects of $\mathcal{G}r\mathsf{C}'$ endowed with the structure of a cocomplete counital differential graded B^+A-B^+A''-bicomodule.

Its image consists of objects which are almost cofree.

2.5.2 Derived category of A_∞-bimodules

Let A and A'' be two augmented A_∞-algebras in C and C''. In this section, we define the derived category of strictly unital A-A''-bipolydules, then we give several descriptions of it.

Model category structure on $\mathsf{Comc}(B^+A, B^+A'')$

Let $(B^+A)^{op}$ be the opposite coalgebra of B^+A defined using the braiding of C. The object $(B^+A)^{op} \otimes B^+A''$ of $\mathsf{C} \otimes \mathsf{C}''$ is a cocomplete differential graded coalgebra. Note that it is not cotensorial in general. The category $\mathsf{Comc}((B^+A)^{op} \otimes B^+A'')$ is equipped with its canonical model category structure (2.2.2.4). The category $\mathsf{Comc}(B^+A, B^+A'')$ becomes a model category thanks to the isomorphism of categories

$$\mathsf{Comc}(B^+A, B^+A'') \to \mathsf{Comc}((B^+A)^{op} \otimes B^+A'').$$

We are now going to show that the fibrant objects of $\mathsf{Comc}(B^+A, B^+A'')$ are exactly the direct factors of almost cofree objects.

An acyclic twisting cochain

Let $(UA)^{op}$ be the opposite algebra of UA defined using the braiding of C. The object $(UA)^{op} \otimes UA''$ of $\mathsf{C} \otimes \mathsf{C}''$ is a differential graded algebra. Equip the category $\mathsf{Mod}((UA)^{op} \otimes UA'')$ with the model category structure of Theorem 2.2.2.1. Let $\mathsf{Mod}(UA, UA'')$ be the *category* of unital differential graded bimodules. The category $\mathsf{Mod}(UA, UA'')$ becomes a model category thanks to the isomorphism of categories

$$\mathsf{Mod}(UA, UA'') \to \mathsf{Mod}((UA)^{op} \otimes UA'').$$

We will construct an admissible acyclic twisting cochain

$$\tau : (B^+A)^{op} \otimes B^+A'' \to (UA)^{op} \otimes UA''.$$

It will follow (2.2.2.3) that the pair of adjoint functors associated with τ (see 2.2.1)

$$(L, R) : \mathsf{Comc}\left((B^+A)^{op} \otimes B^+A''\right) \to \mathsf{Mod}\left((UA)^{op} \otimes UA''\right)$$

is a Quillen equivalence.

The universal twisting cochain (2.2.1.5)

$$\tau_{B^+A} : B^+A \to \Omega^+B^+A = UA$$

induces a twisting cochain

$$\tau'_{B^+A} : (B^+A)^{op} \to (UA)^{op}.$$

We check that

$$\tau = \tau_{B^+A} \otimes \eta \circ \eta + \eta \circ \eta \otimes \tau_{B^+A''} : (B^+A)^{op} \otimes B^+A'' \to (UA)^{op} \otimes UA'',$$

where the symbols η denote the (co)units of B^+A, B^+A'', UA and UA'', is an admissible twisting cochain. By the criterion of acyclicity of twisting cochains (2.2.4.1), the object of $\mathsf{C} \otimes \mathsf{C}''$

$$\begin{aligned}\Big((B^+A)^{op} \otimes B^+A''\Big) \otimes_\tau \Big((UA)^{op} \otimes UA''\Big) = \\ \Big((B^+A)^{op} \otimes_{\tau'_{B^+A}} (UA)^{op} \otimes (B^+A'')^{op} \otimes_{\tau_{B^+A''}} UA''\Big)\end{aligned}$$

is quasi-isomorphic to $e_{\mathsf{C}} \otimes e_{\mathsf{C}''} = e_{\mathsf{C} \otimes \mathsf{C}''}$. The twisting cochain τ is therefore acyclic.

Fibrant objects of $\mathsf{Comc}\left((B^+A)^{op} \otimes B^+A''\right)$

As in the case of polydules over an augmented A_∞-algebra (see 2.4.1.4), we show thanks to obstruction theory (B.3) that the category of A-A''-bipolydules is endowed with the structure of a "model category without limits": weak equivalences, cofibrations and fibrations are defined in the same way as in the case of A-polydules (2.4.1.4). By the same reasoning as that of the proof of Proposition 2.4.1.3, we show that the fibrant objects of the model category $\mathsf{Comc}(B^+A, B^+A'')$ are exactly the direct factors of the almost cofree comodules.

Derived category

The bar construction

$$B : \mathsf{Mod}_\infty(A, A'') \to \mathsf{Comc}(B^+A, B^+A'')$$

is a fully faithful functor. The closure by retracts of its image is the subcategory of fibrant and cofibrant objects. Proposition A.13 and the compatibility of the bar construction with homotopy and weak equivalences shows that the following definition makes sense.

Definition 2.5.2.1. The *category* $\mathcal{H}_\infty(A, A'')$ is the category $\mathsf{Mod}_\infty(A, A'')/\sim$, where $\sim$ is the homotopy relation. The *derived category* $\mathcal{D}_\infty(A, A'')$ is the localization of the category $\mathsf{Mod}_\infty(A, A'')$ with respect to A_∞-quasi-isomorphisms.

By proposition A.13, we have an isomorphism

$$\mathcal{H}_\infty(A, A'') \to \mathcal{D}_\infty(A, A'').$$

We have a fully faithful functor

$$I : \mathsf{Mod}(UA, UA'') \to \mathsf{Mod}_\infty^{\mathsf{strict}}(A, A''), \quad M \to S^{-1}M,$$

where $\mathsf{Mod}_\infty^{\mathsf{strict}}(A, A'')$ is the category of strictly unital A-A''-polydules whose morphisms are the strict A_∞-morphisms. The image of this functor consists of the A-A''-bipolydules M whose morphisms

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A''^{\otimes j} \to M, \quad i, j \geq 0,$$

are zero if the two integers i and j are different from 0. Recall that the analogous functor in the case of polydules is an isomorphism (2.3.4.4).

Lemma 2.5.2.2. The composition of functors

$$J : \mathsf{Mod}(UA, UA'') \xrightarrow{I} \mathsf{Mod}_\infty^{\mathsf{strict}}(A, A'') \hookrightarrow \mathsf{Mod}_\infty(A, A'')$$

induces an equivalence $\mathcal{D}(UA, UA'') \to \mathcal{D}_\infty(A, A'')$.

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathsf{Mod}(UA, UA'') \arrow[r, "I"] \arrow[d, "R"'] & \mathsf{Mod}_\infty^{\mathsf{strict}}(A, A'') \arrow[d, hook] \\ \mathsf{Comc}(B^+A, B^+A'') & \mathsf{Mod}_\infty(A, A'') \arrow[l, "B"] \end{tikzcd}$$

where R and B induce equivalences in the derived categories. This shows that the functor induced by J is fully faithful. Let us show that it is essentially surjective. Let M be an A-A''-bipolydule. The adjunction morphism

$$BM \to RLBM = B^+A \otimes_{\tau_{B^+A}} UA \otimes_{\tau_{B^+A}} BM \otimes_{\tau_{B^+A''}} UA'' \otimes_{\tau_{B^+A''}} B^+A''$$

is a weak equivalence. The bicomodule $RLBM$ is the bar construction of the A-A''-polydule

$$M' = S^{-1}\bigl(UA \otimes_{\tau_{B^+A}} BM \otimes_{\tau_{B^+A''}} UA''\bigr).$$

We then have an A_∞-quasi-isomorphism of A-A''-bipolydules

$$M \to M'$$

and, since M' is in the image of J, we have the result. $\square$

Chapter 3

Units up to homotopy and strict units

Introduction

The A_∞-spaces of [Sta63a] have strict units. In the algebraic framework, the corresponding notion has been defined in (Definition 2.3.2.1).

When A is a strictly unital A_∞-algebra, some properties of unital associative algebras can be generalized to A. For example, we will show the analogue of the isomorphism

$$M \otimes_B B \to M,$$

when B is a unital associative algebra and M a unital B-module (see the generalization in Lemma 4.1.1.6 in chapter 4). However, the A_∞-algebras (in fact A_∞-categories) occurring in geometry [Fuk93] are not strictly unital but *homologically unital*, i.e. H^*A endowed with the multiplication induced by m_2 is a unital graded algebra. The purpose of this chapter is to show that from a homotopy point of view, there is no difference between strict units and homological units. More precisely, we will show that *the subcategory of homologically unital* A_∞*-algebras whose morphisms are the homologically unital* A_∞*-morphisms and the subcategory of strictly unital* A_∞*-algebras whose the morphisms are the* A_∞*-strictly unital morphisms become equivalent after passing to homotopy* (Corollary 3.2.4.4).

Chapter Plan

This chapter is divided into three sections. In section 3.1, we define homological units relative to A_∞-structures. In section 3.2, we show the result stated above. In section 3.3, we compare the different types of compatibility to units of (bi)polydules.

3.1 Definitions

Let C be a base category such as in chapter 1. Let A be an A_∞-algebra over C and let

$$\mu : H^*A \otimes H^*A \to H^*A$$

the morphismed induced by m_2.

Definition 3.1.0.1. A morphism $\eta^A : e \to A$ in $\mathcal{G}r\mathsf{C}$ is a *homological unit* if $m_1 \circ \eta = 0$ and if it induces a unit for the graded associative algebra (H^*A, μ). If A is endowed with a homological unit, we will say that it is *homologically unital*. If A and A' are two homologically unital A_∞-algebras, an A_∞-morphism $f : A \to A'$ is *homologically unital* if f_1 induces a unital morphism

$$H^*A \xrightarrow{\sim} H^*A'.$$

Remark 3.1.0.2. The unit $e \to A$ of a strictly unital A_∞-algebra (Definition 2.3.2.1) is clearly a homological unit. A strictly unital morphism of strictly unital A_∞-algebra is homologically unital.

We find in the works of K. Fukaya [FOOO01] and V. Lyubashenko [Lyu02] other elevations of the notion of unitality. An A_∞-algebra endowed with a "homotopic unit" (defined in [FOOO01] using higher homotopies, see also [Fuk01b]) gives a "A_∞-unital algebra" in the sense of [Lyu02]. The lifting of the notation of unitality due to V. Lyubashenko [Lyu02] specializes to our notion of homological unitality if we work on over a field (V. Lyubashenko works over any commutative ring). Note that homological unitality is not of the type *"up to homotopy"*: it is not defined using higher homotopies satisfying coherence conditions. It is however a valid notion since (as we will see in this chapter) the localization of the category of homologically unital A_∞-algebras with respect to A_∞-quasi-isomorphisms is equivalent to the localization of the category of unital algebras with respect to quasi-isomorphisms.

Definition 3.1.0.3. If f and f' are two homotopically unital morphisms $A \to A'$, a homotopy h between f and f' is *strictly unital* if

$$h_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes l}) = 0, \quad i \geq 1 \text{ and } j + 1 + l = i.$$

Remark 3.1.0.4. If A is a homologically unital A_∞-algebra and H^*A is a minimal model for A (Corollary 1.4.1.4), the homological unit η^A induces a homological unit $\eta^{H^*A} : e \to H^*A$ which additionally satisfies

$$m_2^{H^*A}(\eta^{H^*A} \otimes \mathbf{1}) = m_2^{H^*A}(\mathbf{1} \otimes \eta^{H^*A}) = \mathbf{1}.$$

Let $f : A \to A'$ be a homologically unital morphism and let H^*A and H^*A' be minimal models of A and A'. We recall (Corollary 1.4.1.4) that there exist A_∞-quasi-isomorphisms

$$i : H^*A \to A \quad \text{and} \quad i' : H^*A' \to A'.$$

By point b of Corollary 1.3.1.3, there exists a homotopy inverse p' of i'. The morphism $g = p' \circ f_1 \circ i$ further satisfies $g_1 \eta_{H^*A} = \eta_{H^*A'}$.

3.2 Homologically unital A_∞-algebras

This section is divided into four subsections.

In subsection 3.2.1, we give two proofs of the fact that any homologically unital minimal A_∞-algebra is isomorphic to a strictly unital A_∞-algebra. The first of these proofs is inspired by the theory of deformations of graded algebras and is only valid in characteristic zero. The second is based on the obstruction theory of minimal A_∞-algebras (see appendix B.4).

In the subsections 3.2.2 and 3.2.3, we show, with the help of obstruction theory, that we can make strictly unital any homologically unital A_∞-morphism between strictly unital A_∞-algebras and any homotopy between A_∞-morphisms.

In subsection 3.2.4, we show that every strictly unital A_∞-algebra A admits a strictly unital minimal model A' and strictly unital A_∞-quasi-isomorphisms

$$A' \to A \quad \text{and} \quad A \to A'.$$

We will deduce from this result and from the previous subsections the main result of this chapter (3.2.4.4) : the category $\left(\mathsf{Alg}_\infty\right)_{hu}$ of homologically unital A_∞-algebras whose morphisms are the homologically unital A_∞-morphisms and its nonfull subcategory $\left(\mathsf{Alg}_\infty\right)_{su}$ of the strictly unital A_∞-algebras whose morphisms are the strictly unital A_∞-morphisms, are equivalent up to homotopy.

3.2.1 Unital strictification of A_∞-algebras

Theorem 3.2.1.1 (A. Lazarev [Laz02], P. Seidel [Sei])**.** Any homologically unital minimal A_∞-algebra is isomorphic to a strictly unital minimal A_∞-algebra.

The theorem has been independently proved by P. Seidel [Sei], who uses the same method as us, as well as by A. Lazarev [Laz02]. Our first proof will use deformations and is valid only in characteristic zero. It gives us the existence of strictly unital minimal A_∞-algebra. The second proof is based on the obstruction lemmas of Appendix B.4. It specifies the possible choices of the strictly unital minimal A_∞-algebra.

The two proofs are linked: for a given m_2, the Hochschild complex $C^*(A, A)$ (see Appendix B.4) controls the obstruction to the construction by recurrence of the m_i, $i \geq 3$, of a minimal A_∞-algebra structure on A and it is also the differential graded Lie algebra which describes the problem of deformations of the algebra (A, m_2). We we refer to the articles [SS85] and [KS00] concerning this point.

Corollary 3.2.1.2. Any homologically unital A_∞-algebra is homotopically equivalent to a strictly unital A_∞-algebra.

Proof. Let A be a homologically unital A_∞-algebra and let A' be a minimal model of A. We know that A and A' are homotopically equivalent. The result is then deduced from Theorem 3.2.1.1 applied to A'. □

Remark 3.2.1.3. We will show at the end of this chapter (Proposition 3.2.4.1) that any strictly unital A_∞-algebra A admits a strictly unital minimal model A' such that the A_∞-quasi-morphism

$$A' \to A$$

is strictly unital.

First proof of theorem 3.2.1.1:

Reminders on deformations

Suppose the characteristic of $\mathbb{K}$ is zero. Let $(\mathfrak{g}, \delta, [-,-])$ be a nilpotent differential graded Lie $\mathbb{K}$-algebra, i.e. there exists an integer $N \geq 1$ such that

$$\mathsf{ad}X_1 \mathsf{ad}X_2 \ldots \mathsf{ad}X_N = 0, \quad X_1, \ldots, X_N \in \mathfrak{g}.$$

We denote by $\mathsf{MC}(\mathfrak{g})$ the elements $X \in \mathfrak{g}$ of degree $+1$ which are solutions of the Maurer-Cartan equation

$$\delta(X) + \frac{1}{2}[X, X] = 0.$$

Let Γ be the nilpotent group associated to $\mathfrak{g}^0$. It acts on $\mathfrak{g}^1$ by affine transformations, that is, by the exponentiation of the action of its Lie algebra

$$g.x = \delta(g) + [g, x], \quad g \in \mathfrak{g}^0, x \in \mathfrak{g}^1.$$

This action preserves $\mathsf{MC}(\mathfrak{g})$ and we have the set

$$\mathsf{MC}(\mathfrak{g})/\!\sim\, = \mathsf{MC}(\mathfrak{g})/\Gamma.$$

We recall [GM90] the following result.

Theorem 3.2.1.4. If $\mathfrak{h}$ is a nilpotent differential graded Lie algebra, a homotopy equivalence $f : \mathfrak{h} \to \mathfrak{g}$ induces a bijection

$$\mathsf{MC}(\mathfrak{h})/\!\sim\; \xrightarrow{\sim} \mathsf{MC}(\mathfrak{g})/\!\sim.$$

□

If $\mathfrak{g}'$ is a pronilpotent Lie algebra (i.e. it is the limit of nilpotent algebras $\mathfrak{g}_i$, $i \geq 0$) we define

$$\mathsf{MC}(\mathfrak{g}') = \lim \mathsf{MC}(\mathfrak{g}_i) \quad \text{and} \quad \mathsf{MC}(\mathfrak{g}')/\!\sim\; = \lim \Big(\mathsf{MC}(\mathfrak{g}_i)/\Gamma_i\Big).$$

Link with A_∞-algebras

Let (A, μ) be a unital associative graded $\mathbb{K}$-algebra. The map

$$(D, D') \mapsto [D, D'] = D \circ D' - (-1)^{pq} D' \circ D,$$

where D and D' are homogeneous of degree p and q, endows the complex $(\mathsf{coder}(BA)^+, \delta)$ with a differential graded Lie algebra structure. Let LA denote this Lie algebra. We have an isomorphism of complexes

$$LA \to SC(A, A),$$

where $C(A, A)$ is the Hochschild complex (see Appendix B.4). It sends the Lie bracket of LA to the Gerstenhaber bracket [Ger63]. Let $L^{\geq n}A \subset LA$, $n \geq 3$ be the Lie subalgebra

$$S\Big(\prod_{i \geq n} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes i}, A)\Big).$$

The subalgebras $L^{\geq n}A$, $n \geq 4$, are ideals of $L^{\geq 3}A$ and we have

$$L^{\geq 3}A = \lim_{n \geq 4} \mathfrak{g}_n,$$

where $\mathfrak{g}_n$ is the algebra $L^{\geq 3}A/L^{\geq n}A$. As we have

$$[L^{\geq n}A, L^{\geq n'}A] \subset L^{\geq n+n'-1}A, \quad n, n' \geq 1,$$

the Lie algebras $\mathfrak{g}_n$ are nilpotent and $L^{\geq 3}A$ is pronilpotent. The reduced subcomplex $S\overline{C}(A,A)$ is a Lie subalgebra of LA for the Gerstenhaber bracket. We denote it $\overline{L}A$. Recall that the inclusion $\overline{L}A \hookrightarrow LA$ is a homotopy equivalence (see [CE99, Chap. IX]). By Theorem 3.2.1.4, we have a bijection

$$\Theta : \mathsf{MC}(\overline{L}^{\geq 3}A)/\sim \xrightarrow{\sim} \mathsf{MC}(L^{\geq 3}A)/\sim\,,$$

where $\overline{L}^{\geq 3}A = \overline{L}A \cap L^{\geq 3}A$. An element $b' \in L^{\geq 3}A$ is in $\mathsf{MC}(L^{\geq 3}A)$ if and only if $b = b' + b_2$ (where b_2 corresponds to $m_2 = \mu$) is a differential of $(BA)^+$. In other words, we have a bijection between $\mathsf{MC}(L^{\geq 3}A)$ and the set of minimal A$_\infty$-algebra structures on A whose multiplication m_2 is equal to μ. Under this bijection, the equivalence classes of $\mathsf{MC}(L^{\geq 3}A)$ correspond to the isomorphism classes of A$_\infty$ minimal structures such that m_2 equals μ. Note that an element $b'' \in \mathsf{MC}(L^{\geq 3}A)$ belongs to the subalgebra $\overline{L}^{\geq 3}A$ if and only if the A$_\infty$-structure corresponding to b'' is strictly unital over A. We then deduce from the bijection Θ that any A$_\infty$-structure (whose m_2 equals μ) homologically unital on A is isomorphic to a strictly unital A$_\infty$-structure.

Second proof of theorem 3.2.1.1:

The characteristic of $\mathbb{K}$ is arbitrary.

Lemma 3.2.1.5. Let A be a minimal A$_\infty$-algebra. Let n be an integer ≥ 2 and

$$f_n : A^{\otimes n} \to A$$

a graded morphism of degree $1-n$. There exists a minimal A$_\infty$-algebra A', A$_\infty$-isomorphic to A, whose underlying graded object is A and whose multiplications m'_i, $i \geq 2$, are such that

$$m'_i = m_i \quad \text{if} \quad i \leq n \quad \text{and} \quad m'_{n+1} = m_{n+1} + \delta_{Hoch}(f_n).$$

Proof. Let F be the morphism of graded coalgebras

$$F : BA \to BA$$

determined by the sequence

$$(\mathbf{1}_{SA}, 0, \ldots, 0, F_n, 0 \ldots),$$

where F_n is given by the bijection $F_n \leftrightarrow f_n$ from section 1.2.2. The morphism F is an isomorphism. Let us set

$$b' = F \circ b^A \circ F^{-1}.$$

This is a differential on $\overline{T^c}SA$. The coalgebra $(\overline{T^c}SA, b')$ is thus the bar construction of an A$_\infty$-algebra A', A$_\infty$-isomorphic to A, whose underlying graded object is A. It remains to verify the conditions on the multiplications. The matrix of the morphism of graded coalgebras

$$F : \overline{T^c}(SA) = \bigoplus_{p\geq 1} (SA)^{\otimes p} \xrightarrow{\sim} \overline{T^c}(SA) = \bigoplus_{q \geq 1} (SA)^{\otimes q}$$

is upper triangular and its diagonal consists of identities. The matrix of F^{-1} is thus of the same form. Furthermore, the restriction of F to

$$\overline{T^c_{[n-1]}}SA = \bigoplus_{1 \leq p \leq n-1} (SA)^p$$

is the identity. The same holds for its inverse. The matrix of the differential b^A is strictly upper triangular since b_1^A is zero. A calculation then shows that

$$b'_i = F_1 b_i^A (F^{-1})_1, \quad \text{for} \quad i \leq n,$$

$$b'_{n+1} = F_1 b_{n+1}^A (F^{-1})_1 + F_1 b_2^A (F^{-1})_n + F_n b_2^A (F^{-1})_1.$$

We can deduce the result from the equalities

$$(F^{-1})_n = -F_n \quad \text{and} \quad F_1 = F_1^{-1} = \mathbf{1}_{SA}.$$

□

Proof of Theorem 3.2.1.1. We reason by induction on n. Let $n \geq 2$. Suppose that A is an A_∞-algebra such that, for all $3 \leq i \leq n$, we have

$$m_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes k}) = 0, \quad j + k = n.$$

This is equivalent to requiring that the m_i, $3 \leq i \leq n$, be elements of the reduced Hochschild subcomplex $\overline{C}(A, A)$ (see Section B.4). Let us show that we can construct an A_∞-algebra A', A_∞-isomorphic to A, whose underlying graded object is A and whose multiplications m'_i, $3 \leq i \leq n+1$, are elements of $\overline{C}(A, A)$. By hypothesis on the m_i, $3 \leq i \leq n$, the Hochschild cycle $r(m_3, \cdots, m_{n-1})$ from Lemma B.4.1 belongs to $\overline{C}(A, A)$. As A is an A_∞-algebra, we know from Lemma B.4.1 that

$$\delta_{Hoch}(m_{n+1}) + r(m_3, \cdots, m_n) = 0$$

and that the element $r(m_3, \cdots, m_n)$ is a Hochschild cycle. Thus, the element

$$(m_{n+1}, sr(m_3, \cdots, m_n))$$

from the cone C over the inclusion $\overline{C}(A, A) \hookrightarrow C(A, A)$ is a cycle. Since C is acyclic, this element is the boundary of an element (f_n, sm'_{n+1}). In other words, there exist elements

$$m'_{n+1} \in \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(\overline{A}^{\otimes n+1}, A) \quad \text{and} \quad f_n \in \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes n}, A)$$

such that

$$\delta_{Hoch}(f_n) + m'_{n+1} = m_n \quad \text{and} \quad \delta_{Hoch}(m'_{n+1}) + r(m_3, \cdots, m_n) = 0.$$

By the previous lemma applied to the A_∞-algebra A and to the morphism $-f_n$, there exists an A_∞-algebra A', A_∞-isomorphic to A, such that we have, for all $3 \leq i \leq n+1$,

$$m'_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes k}) = 0, \quad j + k = n.$$

□

3.2.2 Unital strictification of A_∞-morphisms

Theorem 3.2.2.1. A morphism of strictly unital minimal A_∞-algebras which is homologically unital is homotopic to a strictly unital morphism.

Lemma 3.2.2.2. Let A and A' be two minimal A_∞-algebras and $f : A \to A'$ an A_∞-morphism. Let n be an integer ≥ 2 and

$$h_n : A^{\otimes n} \to A$$

a graded morphism of degree $-n$. There exists an A_∞-morphism $f' : A \to A'$ homotopic to f such that

$$f'_i = f_i \quad \text{if} \quad i \leq n \quad \text{and} \quad f'_{n+1} = f_{n+1} - \delta_{Hoch}(h_n).$$

Proof. We are going to construct a morphism f' such that the following

$$(0, \dots, 0, h_n, 0, \dots)$$

defines a homotopy h between f and f'. We construct f'_i by induction on i. Let $i \geq 1$. Suppose there is an A_i-morphism $f' : A \to A'$ such that h defines a homotopy between f and f' that is an A_i-morphism. Set

$$\begin{aligned} f'_{i+1} = f_{i+1} &- \sum (-1)^s m_{r+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes f'_{j_1} \otimes \cdots \otimes f'_{i_t}) \\ &- \sum (-1)^{jk+l} h_z(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}), \end{aligned}$$

where s is the sign appearing in Definition 1.2.1.7. By construction, the sequence

$$(f'_1, \dots, f'_i, f'_{i+1})$$

defines an A_{i+1}-morphism homotopic to f. The morphism f' thus constructed clearly satisfies the desired conditions on the f'_i, for $1 \leq i \leq n+1$. □

Proof of Theorem 3.2.2.1. Let A and A' be two strictly unital minimal A_∞-algebras and

$$f : A \to A'$$

a homologically unital A_∞-morphism. We are looking for a morphism f' homotopic to f such that the morphisms f'_i, $i \geq 1$ satisfy

$$f'_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes l}) = 0, \quad i \geq 2 \text{ and } j+1+l = i.$$

Let us construct f'_i, $1 \leq i \leq n$, by induction on n. Let $n \geq 1$. Suppose that we have a morphism f, such that the morphisms f_i, $2 \leq i \leq n$ satisfy the aforementioned condition. Using the same arguments as in Theorem 3.2.1.1, where we replace the complex $C(A, A)$ by the complex $C(A, A')$ and the obstruction lemma B.4.1 with Lemma B.4.2, we find that there are two elements

$$f'_{n+1} \in \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(\overline{A}^{\otimes n+1}, A') \quad \text{and} \quad h_n \in \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes n}, A')$$

such that

$$\delta_{Hoch}(h_n) + f'_{n+1} = f_n \quad \text{and} \quad \delta_{Hoch}(f'_{n+1}) + r(f_2, \cdots, f_n) = 0.$$

By applying Lemma 3.2.2.2 to f and h_n, there exists a morphism f' homotopic to f such that the morphisms f_i for $2 \leq i \leq n+1$ satisfy the equations

$$f_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes l}) = 0, \quad i \geq 2 \text{ and } j+1+l = i.$$

□

3.2.3 Unital strictification of homotopies

Theorem 3.2.3.1. Let A and A' be two strictly unital minimal A_∞-algebras. If f and g are two strictly unital homotopy A_∞-morphisms $A \to A'$, there exists a strictly unital homotopy between f and g.

Lemma 3.2.3.2. Let A and A' be two minimal A_∞-algebras. Let f and g be two A_∞-homotopic morphisms $A \to A'$ and h a homotopy from f to g. Let $n \geq 2$ and

$$\rho_n : A^{\otimes n} \to A'$$

a graded morphism of degree $-n-1$. There exists a homotopy h' between f and g such that

$$h'_i = h_i \quad \text{if} \quad 1 \leq i \leq n \quad \text{and} \quad h_{n+1} = h'_{n+1} + \delta_{Hoch}(\rho_n).$$

Proof. We proceed as in Lemma 3.2.2.2. Let's denote $F = Bf$, $G = Bg$, and $H = Bh : BA \to BA'$ as the homotopy between F and G. Let R be a (F,G)-coderivation of degree -2 which is given (1.1.2.2) by a sequence

$$(0, \dots, 0, s\rho_n\omega^{\otimes n}, 0, \dots).$$

Consider H' defined by the equality

$$H' = H - b^{A'}R + Rb^{A}.$$

It is a (F, G)-coderivation which is clealry a homotopy beteen F and G. We verify this corresponds to a homotopy h' between f and g such that

$$h'_i = h_i \quad \text{if} \quad 1 \leq i \leq n \quad \text{and} \quad h_{n+1} = h'_{n+1} + \delta_{Hoch}(\rho_n).$$

□

Proof of Theorem 3.2.3.1. We are looking for a homotopy h between f and g such that the morphisms h_i, $i \geq 1$, satisfy

$$h_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes l}) = 0, \quad i \geq 2 \text{ and } j + 1 + l = i.$$

Construct the h_i, $1 \leq i \leq n$, by the recurrence on n. Let $n \geq 1$. Suppose that we have a morphism h, such that the morphisms h_i, $2 \leq i \leq n$, satisfy the aforementioned condition. Using the same arguments as in Theorem 3.2.1.1 where we replace the complex $C(A, A)$ with the complex $C(A, A')$ (see B.4) and the obstruction lemma B.4.1 with the lemma B.4.3, we find that there exist two elements

$$h'_{n+1} \in \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(\overline{A}^{\otimes n+1}, A') \quad \text{and} \quad \rho_n \in \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes n}, A')$$

such that

$$\delta_{Hoch}(\rho_n) + h_{n+1} = h'_n \quad \text{and} \quad \delta_{Hoch}(h'_{n+1}) + r(h_2, \cdots, h_n) = 0.$$

By Lemma 3.2.3.2, there exists a homotopy h' between f and g such that we have the equations

$$h'_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes l}) = 0, \quad i \geq 2 \text{ and } j + 1 + l = i.$$

□

We deduce from Theorems 3.2.1.1, 3.2.2.1 and 3.2.3.1 the following corollary:

Corollary 3.2.3.3. Let A and A' be strictly unital minimal A_∞-algebras and $f : A \to A'$ a strictly unital homotopy equivalence. There exists an inverse up to homotopy g of f which is strictly unital and strictly unital homotopies h and h' between $\mathbf{1}_{A'}$ and $f \circ g$, and between $\mathbf{1}_A$ and $g \circ f$. □

3.2.4 Minimal model of a strictly unital A_∞-algebra

The corollary (3.2.1.2) shows that any homologically unital A_∞-algebra A admits a strictly unital minimal model A' such that the A_∞-quasi-isomorphism

$$f : A' \to A$$

satisfies $f \circ \eta = \eta$. The purpose of this section is to show the following proposition:

Proposition 3.2.4.1. Every strictly unital A_∞-algebra A admits a strictly unital minimal model A' such that the A_∞-quasi-isomorphism

$$f : A' \to A$$

is strictly unital.

Our proof is based on the perturbation lemma (see [HK91], [GS86], [GL89], [GLS91], [Mer99] and [KS01]).

Proof. Let $V = H^*A$. Consider the of complexes morphism $i : (V, 0) \to (A, m_1)$ that induces the identity on homology and such that $i \circ \eta = \eta$. Let $p : A \to K$ be the cokernel of i. The complex K is contractible. The sequence of complexes (i, p) is therefore split. Choose a retraction ρ and a section σ such that

$$\rho \circ \sigma = 0 \quad \text{and} \quad i \circ \rho + \sigma \circ p = \mathbf{1}_A.$$

Let h be a contracting homotopy of K such that $h^2 = 0$. Let $A' = V^\delta$ be the A_∞-algebra (with underlying complex V) and $f = f^\delta$ be the morphism of A_∞-algebras constructed from these data in (1.4.2.1). We aim to show that A' is a strictly unital A_∞-algebra and that the A_∞-morphism f is strictly unital. We will use the notations from the proof of (1.4.2.1). We clearly hvae the equalities

$$m'_1 \circ \eta = 0, \quad m'_2(\eta \otimes \mathbf{1}) = m'_2(\mathbf{1} \otimes \eta) = \mathbf{1} \quad \text{and} \quad f_1 \circ \eta = \eta.$$

It remains to show that the composition of f_i for $i \geq 2$, and m'_i for $i \geq 3$ with

$$\eta_\alpha = (\mathbf{1}^{\otimes \alpha} \otimes \eta \otimes \mathbf{1}^{\otimes i-1-\alpha}), \quad 0 \leq \alpha < i,$$

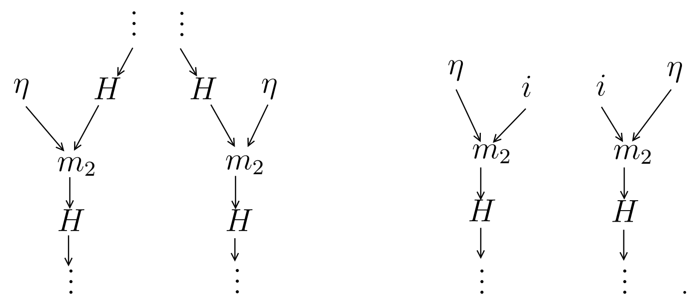
is zero. It suffces to show that the compositions

$$m_{i,T} \circ \eta_\alpha \quad \text{and} \quad f_{i,T} \circ \eta_\alpha, \quad T \in \mathcal{T},$$

are zero. Note that these compositions come from trees $\overline{T}$, colored similaly to $m_{i,T}$ (resp. $f_{i,T}$) except for one leaf which is now colored η. Since A is strictly unital, we have

$$m_j \circ \eta_\beta = 0, \quad j \geq 3, \quad 0 \leq \beta < j.$$

It suffces to verify the nullity of the compositions arising from colored trees where a colored sub-tree has the form



In the first two cases, $m'_{i,T} \circ \eta_\alpha$ and $f_{i,T} \circ \eta_\alpha$ vanish because $H^2 = 0$. In the other cases, they vanish because $i \circ H = 0$. $\square$

Remark 3.2.4.2. One can similarly verify that the morphism q^δ and the homotopy H^δ from the remark (1.4.2.4) are also strictly unital. The perturbation lemma thus produces a contraction in the category of strictly unital A_∞-algebras.

Let $\left(\mathsf{Alg}_\infty\right)_u$ (resp. $\left(\mathsf{Alg}_\infty\right)_{su}$) be the category of strictly unital A_∞-algebras whose morphism spaces consist of homologically unital (resp. strictly unital) morphisms. Let us denote by $\sim_u$ (resp. $\sim_{su}$) the homotopy relation with respect to homotopies in the sense of 1.2.1.7 (resp. to strictly unital homotopies).

Proposition 3.2.4.3. The inclusion

$$\left(\mathsf{Alg}_\infty\right)_{su} \hookrightarrow \left(\mathsf{Alg}_\infty\right)_u$$

induces an equivalence

$$J : \left(\mathsf{Alg}_\infty\right)_{su}/{\sim_{su}} \to \left(\mathsf{Alg}_\infty\right)_u/{\sim_u}\,.$$

Proof. The remark (3.2.4.2) shows that it suffices to show that J induces an isomorphism in the morphism spaces whose source and target are strictly unital minimal A_∞-algebras. We strictify the A_∞-morphisms, then the homotopies between strictly unital A_∞-morphisms using to the theorems (3.2.2.1) and (3.2.3.1). $\square$

Corollary 3.2.4.4. The subcategory $\left(\mathsf{Alg}_\infty\right)_{hu} \subset \mathsf{Alg}_\infty$ of homologically unital A_∞-algebras, whose morphisms are the homologically unital A_∞-morphisms, and the category $\left(\mathsf{Alg}_\infty\right)_{su}$ become equivalent after passing to homotopy. $\square$

Strictly unital trivial (co)fibrations

We finish this section with results that will be useful in Section (4.1.3).

Lemma 3.2.4.5. Let A and A' be strictly unital A_∞-algebras.

a. Let $i : A \to A'$ be a strictly unital trivial cofibration. There exists a strictly unital A_∞-morphism $p : A' \to A$ such that $p \circ i = \mathbf{1}_A$.

b. Let $q : A' \to A$ be a strictly unital trivial fibration. There exists a strictly unital A_∞-morphism $j : A \to A'$ such that $q \circ j = \mathbf{1}_A$.

Proof. The arguments in the proof of the two points are dual, so we will only prove point *a*. Suppose we are given a strictly unital A_∞-morphism p' such that the composition $\alpha = p' \circ i$ is an automorphism of A. Since α is the composition of strictly unital A_∞-morphisms, it is strictly unital. The lemma (3.2.4.6) below shows that the A_∞-morphism α^{-1} is also strictly unital. Let $p = \alpha^{-1} \circ p'$ and we have the result because $p \circ i = \mathbf{1}_A$.

We therefore need to find a strictly unital A_∞-morphism p' such that $p' \circ i$ is an automorphism of A.

First case: the unit η is a boundary of A'.
In this situation, the unit is zero in the cohomology. It follows that A and A' are weakly equivalent to 0. Let us define p'_1 as a retraction of i_1. It satisfies the equality $p'_1 \circ \eta = \eta$. The morphisms p'_i, for $i \geq 2$, are defined by recursion on i. Let h be a contracting homotopy of A. Let us set

$$p'_i = -h \circ r(p'_1, \ldots, p'_{i-1}), \quad i \geq 2,$$

where $r(p'_1, \ldots, p'_{i-1})$ is the cycle from Lemma B.1.5. We verify (by induction) that $r(p'_1, \ldots, p'_{i-1})$ composed with

$$\mathbf{1}^{\otimes \alpha} \otimes \eta \otimes \mathbf{1}^{\otimes \beta}, \quad \alpha + 1 + \beta = i + 1,$$

is zero. The morphisms p'_i, for $i \geq 1$, thus constructed define a well-defined A_∞-morphism due to Lemma (B.1.5). It is strictly unital and, since we have the equality

$$(p' \circ i)_1 = p'_1 \circ i_1 = \mathbf{1}$$

$p' \circ i$ is an automorphism of A.

Second case : the unit η is not a boundary of A'.
Since i is a trivial cofibration, axiom (CM4) of the category Alg_∞ (see 1.3.3.1) gives us an A_∞-morphism $q : A' \to A$ such that $q \circ i = \mathbf{1}_A$. The A_∞-morphism q is clearly homologically unital and satisfies the equality $q_1 \circ \eta = \eta$. Since A and A' are strictly unital, there exists (3.2.4.3) a strictly unital A_∞-morphism $q' : A' \to A$ homotopic to q. Since the unit η is not a boundary of A', there exists a retraction of complexes from $\eta : e \to A'$. This induces a retraction $A' = e \oplus \overline{A}'$. We know that the morphism $q_1 - q'_1$ is homotopic to zero and vanishes on e. It factors through $z \circ t$, where t is the projection $A' \to \overline{A}'$. Since this projection is split in the category of complexes, z is homotopic to zero. Thus, there exists a homotopy h_1 between q_1 and q'_1 such that $h_1 \circ \eta = 0$, and we have the equality $q'_1 \circ i_1 = \mathbf{1}_A + \delta(h_1) \circ i_1$.

Construct the morphisms p'_i, for $i \geq 1$, from the morphisms q'_j, $j \geq 1$, by induction on i: Let

$$p'_1 = q'_1 - \delta(h_1)$$

and, for $i \geq 2$,

$$p'_i = q'_i - \sum (-1)^s m_{r+1+t}(p'_{i_1} \otimes \ldots \otimes p'_{i_r} \otimes h_1 \otimes q'_{j_1} \otimes \ldots \otimes q'_{i_t}) + \sum h_1 \circ m_i,$$

where s is defined in (1.2.1.7). The morphisms p'_i, for $i \geq 1$, define a strictly unital A_∞-morphism $A' \to A$ such that the sequence

$$(h_1, 0, \ldots)$$

is a homotopy between q' and p'. The composition $p' \circ i$ is an automorphism because

$$(p' \circ i)_1 = (q'_1 - \delta(h_1)) \circ i_1 = q'_1 \circ i_1 - \delta(h_1) \circ i_1 = \mathbf{1}_A + \delta(h_1) \circ i_1 - \delta(h_1) \circ i_1 = \mathbf{1}_A.$$

□

Lemma 3.2.4.6. Let A and A' be two strictly unital A_∞-algebras. Let $\alpha : A \to A'$ be a strictly unital A_∞-isomorphism. The A_∞-morphism $\beta = \alpha^{-1}$ is strictly unital.

Proof. We denote by η the unit of A_∞-algebras. As $\alpha_1 \circ \eta = \eta$, we have the equality $\beta_1 \circ \eta = \eta$. We know that the morphism

$$\alpha_2 \circ (\beta_1 \otimes \beta_1) + \alpha_1 \circ \beta_2 : A'^{\otimes 2} \to A$$

is zero. If we compose it with $\eta \otimes \mathbf{1}$ (resp. $\mathbf{1} \otimes \eta$), we find that

$$\alpha_1 \circ \beta_2(\eta \otimes \mathbf{1}), \quad (\text{resp.} \quad \alpha_1 \circ \beta_2(\mathbf{1} \otimes \eta))$$

is zero. Since α_1 is an isomorphism, this implies that

$$\beta_2(\eta \otimes \mathbf{1}) = 0 \quad \text{and} \quad \beta_2(\mathbf{1} \otimes \eta) = 0.$$

We continue by induction on n. Suppose $\beta_1 \eta = \eta$ and

$$\beta_i(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes k}) = 0, \quad j+1+k = i, \quad 2 \leq i \leq n.$$

We deduce the equality

$$(\alpha \circ \beta)_{n+1}(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes k}) = \alpha_1 \circ \beta_{n+1}(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes k}), \quad j+1+k = n+1.$$

Since the defining term $(\alpha \circ \beta)_{n+1}$ is zero, we deduce that

$$\beta_{n+1}(\mathbf{1}^{\otimes j} \otimes \eta \otimes \mathbf{1}^{\otimes k}) = 0, \quad j+1+k = n+1.$$

□

3.3 Unital strictification of (bi)polydules

This section deals with the different types of unit compatibility of A_∞-(bi)polydules. Proofs are omitted because they are similar to those in section 3.2.

3.3.1 Homologically unital polydules

Definition 3.3.1.1. Let A be a homologically unital A_∞-algebra. An A-polydule M is *homologically unital* if H^*M is a unital H^*A-module.

If M and M' are two homologically unital A-polydules, an A_∞-morphism $f : M \to M'$ is always *homologically unital*, i.e. f_1 induces a morphism of unital H^*A-modules

$$H^*M \to H^*M'.$$

Let A be a strictly unital A_∞-algebra. A strictly unital A-polydule (Definition 2.3.2.3) is clearly homologically unital.

The results

Let A be a strictly unital minimal A_∞-algebra.

Theorem 3.3.1.2. Any homologically unital minimal A-polydule is isomorphic to a strictly unital A-polydule. □

Corollary 3.3.1.3. Any homologically unital A-polydule is homotopically equivalent to a strictly unital A-polydule. □

Theorem 3.3.1.4. Let M and M' be two strictly unital minimal A-polydules. Any A_∞-morphism $f : M \to M'$ is homotopic to a strictly unital A_∞-morphism. □

Theorem 3.3.1.5. Let M and M' be two strictly unital minimal A-polydules. If f and g are two homotopic strictly unital A_∞-morphisms $M \to M'$, there exists a strictly unital homotopy between f and g. □

Corollary 3.3.1.6. Let M and M' be strictly unital A-polydules and $f : M \to M'$ a strictly unital homotopy equivalence. There exists an inverse up to homotopy g of f which is strictly unital and strictly unital homotopies h and h' between $\mathbf{1}_{M'}$ and $f \circ g$, and between $\mathbf{1}_M$ and $g \circ f$. □

Let A be a strictly unital A_∞-algebra.

Proposition 3.3.1.7. Every strictly unital A-polydule M admits a strictly unital minimal model M' such that the A_∞-quasi-isomorphism

$$f : M' \to M$$

is strictly unital. □

Let $\left(\mathsf{Nod}_\infty A\right)_u$ be the full *subcategory* of $\mathsf{Nod}_\infty A$ consisting of strictly unital A-polydules.

Proposition 3.3.1.8. The inclusion

$$\mathsf{Mod}_\infty A \hookrightarrow \left(\mathsf{Nod}_\infty A\right)_u$$

induces an equivalence

$$\mathsf{Mod}_\infty A/\sim \xrightarrow{\sim} \left(\mathsf{Nod}_\infty A\right)_u/\sim,$$

where the symbols $\sim$ denote the homotopy relation (2.3.2.3) and (2.3.1.10). □

3.3.2 Homologically unital bipolydules

Let C and C' be two differential graded coalgebras and let N and N' be two differential graded C-C'-bicomodules. Let Δ^R and Δ^L denote the right and left comultiplication of these bicomodules.

Definition 3.3.2.1. A *coderivation of bicomodules* is a morphism

$$K : N \to N'$$

such that

$$\Delta^L \circ K = (\mathbf{1} \otimes K) \circ \Delta^L \quad \text{and} \quad \Delta^R \circ K = (K \otimes \mathbf{1}) \circ \Delta^R.$$

Let A and A' be two unital associative graded algebras and M a graded A-A'-bimodule. Consider them as A_∞-algebras and as an A-A'-bipolydule. The space of coderivations of B^+A-B^+A'-bicomodules

$$\mathsf{coder}(BM, BM)$$

plays in this section the role of space

$$\mathsf{coder}((BA)^+, (BA)^+)$$

in Section B.4.

Let A and A' be two strictly unital A_∞-algebras. Let $\left(\mathsf{Nod}_\infty(A, A')\right)_u$ the full *subcategory* of $\mathsf{Nod}_\infty(A, A')$ formed of strictly unital A-A'-bipolydules. We show the following proposition in the same way as before:

Proposition 3.3.2.2. The inclusion

$$\mathsf{Mod}_\infty(A, A') \hookrightarrow \left(\mathsf{Nod}_\infty(A, A')\right)_u$$

induces an equivalence

$$\mathsf{Mod}_\infty(A, A')/\sim \xrightarrow{\sim} \left(\mathsf{Nod}_\infty(A, A')\right)_u/\sim\,,$$

where the symbols $\sim$ denote the homotopy relations. $\square$

Chapter 4

Derived category

Introduction

Let A be an *augmented* A_∞-algebra. In the chapter 2, we have shown that the derived category $\mathcal{D}_\infty A$ admits three descriptions:

$$\big(\mathsf{Mod}_\infty A\big)[Qis^{-1}], \quad \mathcal{H}_\infty A = \mathsf{Mod}_\infty A/\!\sim \quad \text{and} \quad \big(\mathsf{Mod}_\infty^{\mathsf{strict}} A\big)[Qis^{-1}]$$

where $\sim$ is the homotopy relation. In this chapter, we define the derived category $\mathcal{D}_\infty A$ of any A_∞-algebra A. We show that the three descriptions above hold if A is *strictly unital*.

Plan of the chapter

Let B, B' be two $\mathbb{K}$-associative algebras and X a B-B'-bimodule. The standard functors associated with X are the adjoint functors

$$\mathsf{Hom}_{B'}(X, -) \quad \text{and} \quad ? \otimes_B X.$$

Now let A and A' be A_∞-algebras and X be an A-A'-bipolydule. In the section 4.1.1, we define the standard functors

$$\overset{\infty}{\mathsf{Hom}}_{A'}(X, -) \quad \text{and} \quad ? \overset{\infty}{\otimes}_A X$$

and we show that they form a pair of adjoint functors.

In section 4.1.2, we define the category $\mathcal{D}_\infty A$ of any A_∞-algebra and we describe it in the case where A is H-unital (Propositions 4.1.2.10). In section 4.1.3, we show (Theorem 4.1.3.1) that if A is strictly unital, the category $\mathcal{D}_\infty A$ as defined in the previous section is equivalent to the categories

$$\big(\mathsf{Mod}_\infty A\big)[Qis^{-1}], \quad \mathcal{H}_\infty A \quad \text{and} \quad \big(\mathsf{Mod}_\infty^{\mathsf{strict}} A\big)[Qis^{-1}].$$

In particular, if A is an augmented A_∞-algebra, the definitions of the derived category from chapter 2 and from this one are equivalent. In section 4.2, we study the derived category $\mathcal{D}_\infty(A, A')$, where A and A' are two A_∞-algebras.

4.1 The derived category of A-infinity modules

4.1.1 The standard functors

Notations

Let C be a bicategory (see [ML98, Chap. XII, §6]). Suppose that, for all $\mathbb{O}, \mathbb{O}' \in \mathsf{Obj}\,\mathsf{C}$, the *category*

$$\mathsf{C}(\mathbb{O}, \mathbb{O}') = \mathsf{Hom}_{\mathsf{C}}(\mathbb{O}, \mathbb{O}')$$

is a semisimple Grothendieck k-category and that the composition functor (associative up to a given isomorphism)

$$\mathsf{C}(\mathbb{O}', \mathbb{O}'') \times \mathsf{C}(\mathbb{O}, \mathbb{O}') \to \mathsf{C}(\mathbb{O}, \mathbb{O}''), \quad (M, N) \mapsto M \circ N,$$

where $\mathbb{O}, \mathbb{O}', \mathbb{O}'' \in \mathsf{Obj}\,\mathsf{C}$, is k-bilinear in morphism spaces. We call this functor the *tensor product over* $\mathbb{O}'$ and denote it

$$M \otimes_{\mathbb{O}'} N = M \circ N.$$

Suppose further that, for any object X of $\mathsf{C}(\mathbb{O}', \mathbb{O}'')$, the functor

$$? \otimes_{\mathbb{O}'} X : \mathsf{C}(\mathbb{O}, \mathbb{O}') \to \mathsf{C}(\mathbb{O}, \mathbb{O}'')$$

admits a right adjoint

$$\mathsf{Hom}_{\mathbb{O}''}(X, ?) : \mathsf{C}(\mathbb{O}, \mathbb{O}'') \to \mathsf{C}(\mathbb{O}, \mathbb{O}').$$

Note that the tensor product over $\mathbb{O}$

$$\mathsf{C}(\mathbb{O}, \mathbb{O}) \times \mathsf{C}(\mathbb{O}, \mathbb{O}) \to \mathsf{C}(\mathbb{O}, \mathbb{O}), \quad (M, N) \mapsto M \otimes_{\mathbb{O}} N,$$

where $\mathbb{O} \in \mathsf{Obj}\,\mathsf{C}$, endows the category $\mathsf{C}(\mathbb{O}, \mathbb{O})$ with a monoidal category structure. Let $e_{\mathbb{O}}$ denote the neutral element for the tensor product. Let $\mathbb{O}', \mathbb{O}''$ be objects of C. The category $\mathsf{C}(\mathbb{O}, \mathbb{O})$ acts on the right on the category $\mathsf{C}(\mathbb{O}', \mathbb{O})$ and on the left on the category $\mathsf{C}(\mathbb{O}, \mathbb{O}')$ by the tensor product $\otimes_{\mathbb{O}}$.

The following example appears naturally in the study of A_∞-categories (Section 5.1.1).

Example 4.1.1.1. The bicategory C has as its objects sets considered as discrete categories. Let $\mathbb{A}$ and $\mathbb{B}$ be two sets. We define $\mathsf{C}(\mathbb{A}, \mathbb{B})$ as the category of functors

$$\mathbb{B}^{op} \times \mathbb{A} \to \mathsf{Vect}\mathbb{K}.$$

The composition of C is given by tensor products over the categories. The adjoint functor

$${}_{\mathbb{A}}?_{\mathbb{B}} \otimes_{\mathbb{B}} ({}_{\mathbb{B}}X_{\mathbb{C}}) : \mathsf{C}(\mathbb{A}, \mathbb{B}) \to \mathsf{C}(\mathbb{A}, \mathbb{C})$$

can be rewritten more naturally as

$$\mathsf{Hom}_{\mathbb{C}}({}_{\mathbb{B}}X_{\mathbb{C}}, {}_{\mathbb{A}}?_{\mathbb{C}}) : \mathsf{C}(\mathbb{A}, \mathbb{C}) \to \mathsf{C}(\mathbb{A}, \mathbb{B}).$$

Section Plan

Let $\mathbb{P}$, $\mathbb{O}$ and $\mathbb{O}'$ be objects of C. Let A and A' be two A_∞-algebras in $\mathsf{C}(\mathbb{O},\mathbb{O})$ and $\mathsf{C}(\mathbb{O}',\mathbb{O}')$ and X an A-A'-bipolydule in $\mathsf{C}(\mathbb{O},\mathbb{O}')$. We are going to construct a pair of adjoint functors

$$\big(? \overset{\infty}{\otimes}_A X, \overset{\infty}{\mathsf{Hom}}_{A'}(X,-)\big) : \mathsf{Nod}_\infty A \to \mathsf{Nod}_\infty A'.$$

where $\mathsf{Nod}_\infty A$ is the category of A-polydules in $\mathsf{C}(\mathbb{P},\mathbb{O})$ and $\mathsf{Mod}_\infty A'$ is the category of A'-polydules in $\mathsf{C}(\mathbb{P},\mathbb{O}')$.

The Functor $\overset{\infty}{\mathsf{Hom}}_{A'}(X,-) : \mathsf{Nod}_\infty A' \to \mathsf{Nod}_\infty A$

Let N' be an A'-polydule. Note that $SX \otimes T^cSA'$ is an object of the category $\mathsf{C}(\mathbb{O},\mathbb{O}')$ and that $SN' \otimes T^cSA'$ is an object of $\mathsf{C}(\mathbb{P},\mathbb{O}')$. We define the graded object of $\mathsf{C}(\mathbb{P},\mathbb{O})$ underlying $\overset{\infty}{\mathsf{Hom}}_{A'}(X,N')$ as

$$\mathsf{Hom}_{\mathsf{Comc}\, T^cSA'}(SX \otimes T^cSA', SN' \otimes T^cSA'),$$

where Hom denotes the adjoint functor $\mathsf{Hom}_{\mathbb{O}'}$. Its differential is the morphism

$$\delta : F \mapsto b^{BN'} \circ F - (-1)^{|F|} F \circ b^{BX_{A'}}$$

where $BX_{A'} = SX \otimes T^cSA'$ is the bar construction of X as an A'^{+}-polydule and where the morphism F has degree $|F|$. This is a differential graded module over the differential graded algebra

$$\mathsf{End}(BX_{A'}) = \Big(\mathsf{Hom}_{\mathcal{G}r T^cSA'}(SX \otimes T^cSA', SX \otimes T^cSA'), \delta\Big).$$

The A-polydule structure is given by the restriction of the differential graded $\mathsf{End}(BX_{A'})$-module $\overset{\infty}{\mathsf{Hom}}_{A'}(X,N')$ along the A_∞-morphism

$$A \to \mathsf{End}(BX_{A'})$$

defined in the key lemma (5.3.0.1). Let us explain this structure. The morphism

$$m_i^{\mathsf{H}} : \overset{\infty}{\mathsf{Hom}}_{A'}(X,N') \otimes A^{\otimes i-1} \to \overset{\infty}{\mathsf{Hom}}_{A'}(X,N'), \quad i \geq 1,$$

is given by the differential of the space if $i = 1$ and, otherwise, by the morphism

$$\begin{array}{c} S\mathsf{Hom}_{\mathsf{Comc}\, T^cSA'}(SX \otimes T^cSA', SN' \otimes T^cSA') \otimes (SA)^{\otimes i-1} \\ \Big\downarrow{\scriptstyle b_i^{\mathsf{H}}} \\ S\mathsf{Hom}_{\mathsf{Comc}\, T^cSA'}(SX \otimes T^cSA', SN' \otimes T^cSA') \end{array}$$

which sends an element $s\Gamma \otimes \phi \in S\mathsf{Hom}_{\mathsf{Comc}\, T^cSA'}(SX \otimes T^cSA', SN' \otimes T^cSA') \otimes (SA)^{\otimes i-1}$ to

$$b_2^{\mathsf{Comc}}(s\Gamma \otimes s\Phi) \in S\mathsf{Hom}_{\mathsf{Comc}\, T^cSA'}(SX \otimes T^cSA', SN' \otimes T^cSA'),$$

where the morphism b_2^{Comc} corresponds to the composition of the category $\mathsf{Comc}\, T^cSA'$ and Φ is defined in the key lemma (5.3.0.1). A morphism $f : N' \to N''$ in $\mathsf{Nod}_\infty A'$ induces a morphism of differential graded $\mathsf{End}(BX_{A'})$-modules

$$F_* : \mathsf{Hom}(SX \otimes T^cSA', SN' \otimes T^cSA') \to \mathsf{Hom}(SX \otimes T^cSA', SN'' \otimes T^cSA'),$$

where F_* is induced by the bar construction F of f. Thus, the morphism F_* is strict as a morphism of A-polydules. This gives us a functor

$$\overset{\infty}{\mathsf{Hom}}_{A'}(X, -) : \mathsf{Nod}_\infty A' \to \mathsf{Nod}_\infty^{\mathsf{strict}} A \hookrightarrow \mathsf{Nod}_\infty A.$$

Remark 4.1.1.2. If A is strictly unital and if X is a strictly unital A-A'-bipolydule for A, i.e., if the composition

$$m_{i,j}(\mathbf{1}^{\odot\alpha} \odot \eta \odot \mathbf{1}^{\odot\beta} \otimes \mathbf{1}_M \otimes \mathbf{1}^{\otimes j}), \quad i, j \geq 0,$$

is zero if $(i,j) \neq (1,0)$, equal to $\mathbf{1}$ otherwise, the A-polydule $\overset{\infty}{\mathsf{Hom}}_{A'}(X, N)$ is strictly unital. We then get a functor

$$\overset{\infty}{\mathsf{Hom}}_{A'}(X, -) : \mathsf{Nod}_\infty A' \to \mathsf{Mod}_\infty^{\mathsf{strict}} A \hookrightarrow \big(\mathsf{Nod}_\infty A\big)_u,$$

where $\big(\mathsf{Nod}_\infty A\big)_u$ is the full subcategory of $\mathsf{Nod}_\infty A$ consisting of strictly unital objects.

The functor $? \overset{\infty}{\otimes}_A X : \mathsf{Nod}_\infty A \to \mathsf{Nod}_\infty A'$

Let N be an A-polydule. The graded object of $\mathsf{C}(\mathbb{P}, \mathbb{O}')$ underlying $N \overset{\infty}{\otimes}_A X$ is

$$N \otimes T^c SA \otimes X.$$

The A'-polydule structure over $N \otimes T^c SA \otimes X$ is given by a differential b over $S(N \otimes T^c SA \otimes X) \otimes T^c SA'$. The suspension of this differential graded $T^c SA'$-comodule is identified with the *cotensor product*

$$(SN \otimes T^c SA) \,\square_{T^c SA} (T^c SA \otimes SX \otimes T^c SA'),$$

i.e., the kernel

$$\mathsf{ker}\big(BN \otimes BX \xrightarrow{\Delta \otimes \mathbf{1} - \mathbf{1} \otimes \Delta^L} BN \otimes T^c SA \otimes BX\big),$$

where $BX = T^c SA \otimes SX \otimes T^c SA'$ is the bar construction of X as a strictly unital A^+-A'^+-bipolydule. A morphism of A-polydules $f : N \to N'$ induces a strict morphism

$$\Big(\omega \circ F \circ s\Big) \otimes \mathbf{1}_X : N \otimes T^c SA \otimes X \to N' \otimes T^c SA \otimes X.$$

We thus obtain a functor

$$? \overset{\infty}{\otimes}_A X : \mathsf{Nod}_\infty A \to \mathsf{Nod}_\infty^{\mathsf{strict}} A' \hookrightarrow \mathsf{Nod}_\infty A'.$$

Remark 4.1.1.3. If A' is strictly unital and if X is a strictly unital A-A'-bipolydule for A', i.e., if the composition

$$m_{i,j}(\mathbf{1}^{\otimes i} \otimes \mathbf{1}_M \otimes \mathbf{1}^{\odot\alpha} \odot \eta \odot \mathbf{1}^{\odot\beta}), \quad i, j \geq 0,$$

is zero if $(i,j) \neq (0,1)$, equal to $\mathbf{1}$ otherwise, the A'-polydule $N \overset{\infty}{\otimes}_A X$ is strictly unital. We then obtain a functor

$$? \overset{\infty}{\otimes}_A X : \mathsf{Nod}_\infty A \to \mathsf{Mod}_\infty^{\mathsf{strict}} A' \hookrightarrow \big(\mathsf{Nod}_\infty A'\big)_u.$$

Lemma 4.1.1.4. The functor $? \overset{\infty}{\otimes}_A X$ is left adjoint to the functor $\overset{\infty}{\mathsf{Hom}}_{A'}(X, ?)$

Proof. Let L be an object of $\mathsf{Nod}_\infty A$ and R an object of $\mathsf{Nod}_\infty A'$. Let there be graded morphisms in $\mathsf{C}(\mathbb{O}', \mathbb{O}')$ of degree $2-i$

$$f_j : L \otimes T^c SA \otimes X \otimes A'^{\otimes j} \to R, \quad j \geq 0.$$

Let F_j, $j \geq 0$ be the morphisms given by the bijections $F_j \leftrightarrow f_j$. They are given by morphisms of degree 0

$$F_{i,j} : SL \otimes (SA)^{\otimes i} \otimes X \otimes (SA')^{\otimes j} \to SR, \quad i, j \geq 0.$$

Let $i \geq 0$. Let g_i be graded morphisms of $\mathsf{C}(\mathbb{P}, \mathbb{O})$ of degree $1-i$

$$g_i : L \otimes A^{\otimes i} \to \mathsf{Hom}_{T^c SA'}(SX \otimes T^c SA', SR \otimes T^c SA')$$

defined by the equation

$$G_i(\lambda \otimes \phi) = s(\Gamma) \in S\mathsf{Hom}_{T^c SA'}(SX \otimes T^c SA', SR \otimes T^c SA')$$

where $\lambda \otimes \phi$ is an element of $SL \otimes (SA)^{\otimes i}$ of degree $r = |\lambda \otimes \phi|$, where G_i is given by the bijections $g_i \leftrightarrow G_i$ and where the morphism Γ is the unique morphism (see 2.1.2.1) such that the composition $p_1 \circ \Gamma$ has as composants the morphisms

$$SX \otimes (SA')^{\otimes j} \xrightarrow{(-1)^{|r|}\lambda \otimes \phi \otimes \mathbf{1}} SN \otimes (SA)^{\otimes i} \otimes SX \otimes (SA')^{\otimes j} \xrightarrow{F'_{i,j}} SR;$$

here the morphism $F'_{i,j}$ is the morphism $F_{i,j}\omega$. We need to show the equivalence between the following two points.

a. The morphisms g_j define an A_∞-morphism of A-polydules

$$L \to \overset{\infty}{\mathsf{Hom}}_{A'}(X, R).$$

b. The morphisms f_j define an A_∞-morphism of A'-polydules

$$L \overset{\infty}{\otimes}_A X \to R.$$

Suppose that the statement a is true: we have the equalities

$$\sum_{k+l+m=n} G_{k+1+m}(\mathbf{1}^{\otimes k} \otimes b_l \otimes \mathbf{1}^{\otimes m}) = \sum_{k+m=n} b^{\mathsf{H}}_{1+m}(G_k \otimes \mathbf{1}^{\otimes m}), \quad n \geq 1,$$

where the symbols b_l must be interpreted appropriately. We will show that this is equivalent to the equations in morphism spaces

$$\mathsf{Hom}_{\mathsf{C}(\mathbb{P},\mathbb{O}')}\Big(S(L \overset{\infty}{\otimes}_A X) \otimes (SA')^{\otimes n-1}, SR\Big), \quad n \geq 0,$$

$$\sum_{k+l+m=t} F_{k+1+m}(\mathbf{1}^{\otimes k} \otimes b_l \otimes \mathbf{1}^{\otimes m}) = \sum_{k+m=t} b_{1+m}(F_k \otimes \mathbf{1}^{\otimes m}), \quad t \geq 1.$$

Let $\lambda \otimes \phi \in SL \otimes (SA)^{\otimes n-1}$ and $\kappa \otimes \phi' \in SX \otimes (SA')^{\otimes t-1}$. We calculate

$$G_{k+1+m}(\mathbf{1}^{\otimes k} \otimes b_l \otimes \mathbf{1}^{\otimes m})(\lambda \otimes \phi)(\kappa \otimes \phi').$$

In the case where $k=0$, we have

$$
\begin{aligned}
& G_{1+m}(b_l^R \otimes \mathbf{1}^{\otimes m})(\lambda \otimes \phi)(\kappa \otimes \phi') \\
= \; & G_{1+m}(b_l^R(\lambda \otimes \phi_{l-1}) \otimes \phi_m)(\kappa \otimes \phi') \\
= \; & s\Gamma(\kappa \otimes \phi') \\
= \; & (-1)^{|\lambda \otimes \phi_{l-1}|+1+|\phi_m|} F'_{1+m,t-1}(b_l^R(\lambda \otimes \phi_{l-1}) \otimes \phi_m \otimes \kappa \otimes \phi') \\
= \; & (-1)^{|\lambda \otimes \phi|+1} F'_{1+m,t-1}(b_l^R \otimes \mathbf{1}^{\otimes m} \otimes \mathbf{1} \otimes \mathbf{1}^{\otimes t-1})(\lambda \otimes \phi_{l-1} \otimes \phi_m \otimes \kappa \otimes \phi') \\
= \; & (-1)^{|\lambda \otimes \phi|+1} F'_{1+m,t-1}(b_l^R \otimes \mathbf{1}^{\otimes m} \otimes \mathbf{1} \otimes \mathbf{1}^{\otimes t-1})(\lambda \otimes \phi \otimes \kappa \otimes \phi'),
\end{aligned}
$$

where $\phi_1 \otimes \phi_2 = \phi$ and in the case where $k \neq 0$, we have

$$
\begin{aligned}
& G_{k+1+m}(\mathbf{1}^{\otimes k} \otimes b_l^A \otimes \mathbf{1}^{\otimes m})(\lambda \otimes \phi)(\kappa \otimes \phi') \\
= \; & (-1)^{|\lambda|+|\phi_{k-1}|} G_{k+1+m}(\lambda \otimes \phi_{k-1} \otimes b_l^A(\phi_l) \otimes \phi_m)(\kappa \otimes \phi') \\
= \; & (-1)^{|\lambda|+|\phi_{k-1}|} s\Gamma(\kappa \otimes \phi') \\
= \; & (-1)^{|\lambda|+|\phi_{k-1}|+|\lambda \otimes \phi|+1} F'_{k+1+m,t-1}(\lambda \otimes \phi_{k-1} \otimes b_l^A(\phi_l) \otimes \phi_m \otimes \kappa \otimes \phi') \\
= \; & (-1)^{|\lambda|+|\phi_{k-1}|+|\lambda \otimes \phi|+1} F'_{k+1+m,t-1}(\mathbf{1}^{\otimes k} \otimes b_l^A \otimes \mathbf{1}^{\otimes m} \otimes \mathbf{1} \otimes \mathbf{1}^{\otimes t-1}) \\
& \qquad (\lambda \otimes \phi \otimes \kappa \otimes \phi') \\
= \; & (-1)^{|\lambda \otimes \phi|+1} F'_{k+1+m,t-1}(\mathbf{1}^{\otimes k} \otimes b_l^A \otimes \mathbf{1}^{\otimes m} \otimes \mathbf{1} \otimes \mathbf{1}^{\otimes t-1})(\lambda \otimes \phi \otimes \kappa \otimes \phi'),
\end{aligned}
$$

where $\phi_1 \otimes \phi_2 \otimes \phi_3 = \phi$. The term

$$
b_1^{\mathsf{H}}(G_n)(\lambda \otimes \phi)(\kappa \otimes \phi')
$$

equals

$$
\begin{aligned}
& b_1^{\mathsf{H}}(G_n(\lambda \otimes \phi))(\kappa \otimes \phi') \\
= \; & b_1^{\mathsf{H}}(s\Gamma)(\kappa \otimes \phi') \\
= \; & -sm_1^{\mathsf{H}}(\Gamma)(\kappa \otimes \phi') \\
= \; & -s[b \circ \Gamma - (-1)^{|\lambda+\phi|+1}\Gamma \circ b](\kappa \otimes \phi') \\
= \; & (b^L \circ s\Gamma)(\kappa \otimes \phi') - (-1)^{|\lambda+\phi|+1}(s\Gamma \circ b^{X_{A'}})(\kappa \otimes \phi') \\
= \; & (-1)^{|\lambda+\phi|+1} \sum b^L_{\beta+1}(F'_{n-1,\alpha}(\lambda \otimes \phi \otimes \kappa \otimes \phi'_\alpha) \otimes \phi'_\beta) \\
& +s\Gamma \sum (\mathbf{I}^{\otimes \gamma_1} \otimes b_{\gamma_2} \otimes \mathbf{1}^{\otimes \gamma_3})(\kappa \otimes \phi'_{\gamma_1} \otimes \phi'_{\gamma_2} \otimes \phi'_{\gamma_3}) \\
= \; & (-1)^{|\lambda+\phi|+1} \sum b^L_{\beta+1}(F'_{n-1,\alpha} \otimes \mathbf{1}^{\otimes \beta})(\lambda \otimes \phi \otimes \kappa \otimes \phi') \\
& \sum F'_{n-1,\gamma_1+\gamma 3}(\lambda \otimes \phi \otimes \mathbf{1}^{\otimes \gamma_1} \otimes b_{\gamma_2} \otimes \mathbf{1}^{\otimes \gamma_3})(\kappa \otimes \phi') \\
= \; & (-1)^{|\lambda+\phi|+1} \sum b^L_{\beta+1}(F'_{n-1,\alpha} \otimes \mathbf{1}^{\otimes \beta})(\lambda \otimes \phi \otimes \kappa \otimes \phi') \\
& +(-1)^{|\lambda+\phi|} \sum F'_{n-1,\gamma_1+\gamma 3}(\mathbf{1} \otimes \mathbf{1}^{\otimes n-1} \otimes \mathbf{1}^{\otimes \gamma_1} \otimes b_{\gamma_2} \otimes \mathbf{1}^{\otimes \gamma_3})(\lambda \otimes \phi \otimes \kappa \otimes \phi'),
\end{aligned}
$$

where $\phi'_\alpha \otimes \phi'_\beta = \phi$ and the indices of the first sum are such that $\alpha + \beta = t-1$, where the indices of the second sum are such that $\gamma_1 + \gamma_2 + \gamma_3 = t$ and the symbols b_{γ_2} must be interpreted according to their place by b^X_{0,γ_2-1} or by $b^{A'}_{\gamma_2}$. The term

$$
b^{\mathsf{H}}_{1+m}(G_k \otimes \mathbf{1}^{\otimes m})(\lambda \otimes \phi)(\kappa \otimes \phi')
$$

equals

$$
\begin{array}{rl}
& b^{\mathsf{H}}_{1+m}(G_k(\lambda\otimes\phi_{k-1})\otimes\phi_m)(\kappa\otimes\phi') \\
= & b^{\mathsf{H}}_{1+m}(s\Gamma_{k-1}\otimes\phi_m)(\kappa\otimes\phi') \\
= & b^{\mathsf{Comc}}_2(s\Gamma_{k-1}\otimes s\Phi_m)(\kappa\otimes\phi') \\
= & (-1)^{|\lambda+\phi_{k-1}|+1}b^{\mathsf{Comc}}_2(s\otimes s)(\Gamma_{k-1}\otimes\Phi_l)(\kappa\otimes\phi') \\
= & (-1)^{|\lambda+\phi_{k-1}|+1}sm^{\mathsf{Comc}}_2(\Gamma_{k-1}\otimes\Phi_l)(\kappa\otimes\phi') \\
= & (-1)^{|\lambda+\phi_{k-1}|+1}(s\Gamma_{k-1}\circ\Phi_l)(\kappa\otimes\phi') \\
= & (-1)^{|\lambda+\phi_{k-1}|+1+|\lambda+\phi_{k-1}|+|\phi_m|}\sum F'_{k,\beta}(\lambda\otimes\phi_{k-1}\otimes b^X_{m,\alpha}(\phi_m\otimes\kappa\otimes\phi'_{m'})\otimes\phi'_\beta) \\
= & (-1)^{1+|\phi_m|}\sum F'_{k,\beta}(\lambda\otimes\phi_{k-1}\otimes b^X_{m,\alpha}(\phi_m\otimes\kappa\otimes\phi'_\alpha)\otimes\phi'_\beta) \\
= & (-1)^{|\lambda|+|\phi|+1}\sum F'_{k,\beta}(\mathbf{1}\otimes\mathbf{1}^{\otimes k-1}\otimes b^X_{m,\alpha}(\mathbf{1}^{\otimes m}\otimes\mathbf{1}\otimes\mathbf{1}^{\otimes\alpha})\otimes\mathbf{1}^{\otimes\beta}) \\
& \qquad (\lambda\otimes\phi\otimes\kappa\otimes\phi'),
\end{array}
$$

where the indices of the sum are such that $\alpha+\beta=t-1$. The equality $F'_{i,j}=F_{i,j}\omega$ gives us the equivalence between the points a and b. □

Remark 4.1.1.5. If A and A' are strictly unital and if X is a strictly unital A-A'-bipolydule, the adjunction

$$(?\overset{\infty}{\otimes}_A X, \overset{\infty}{\mathsf{Hom}}_{A'}(X,?)) : \mathsf{Nod}_\infty A \to \mathsf{Nod}_\infty A'$$

is not restricted to the subcategories $\mathsf{Mod}_\infty A$ and $\mathsf{Mod}_\infty A'$. However, Proposition (3.3.1.8) shows that the restriction functors

$$?\overset{\infty}{\otimes}_A X : \mathsf{Mod}_\infty A \to \mathsf{Mod}_\infty A' \quad\text{and}\quad \overset{\infty}{\mathsf{Hom}}_{A'}(X,?) : \mathsf{Mod}_\infty A' \to \mathsf{Mod}_\infty A$$

induce adjoint functors in the derived categories $\mathcal{D}_\infty A$ and $\mathcal{D}_\infty A'$ (defined in Section 4.1.3).

Let A be a strictly unital A_∞-algebra. Consider A as a strictly unital A-A-bipolydule. Also denote by

$$?\overset{\infty}{\otimes}_A X \quad\text{and}\quad \overset{\infty}{\mathsf{Hom}}_A(X,?)$$

the standard functors restricted to the subcategory $\left(\mathsf{Nod}_\infty A\right)_u$ (see the definition in Proposition 3.3.1.8).

Lemma 4.1.1.6. Consider the category of endofunctors of the category $(\mathsf{Nod}_\infty A)_u$.

a. There is a canonical morphism of functors $?\overset{\infty}{\otimes}_A A \to \mathbf{1}$ which is a quasi-isomorphism.

b. There exists a canonical morphism of functors $\mathbf{1} \to \overset{\infty}{\mathsf{Hom}}_A(A,?)$ which is a quasi-isomorphism.

Proof. The adjunction

$$(?\overset{\infty}{\otimes}_A A, \overset{\infty}{\mathsf{Hom}}_A(A,?)) : \left(\mathsf{Nod}_\infty A\right)_u \to \left(\mathsf{Nod}_\infty A\right)_u,$$

suffices to show point a. Let M be a strictly unital A-polydule. We have an A_∞-morphism of A-polydules

$$g : M\overset{\infty}{\otimes}_A A \to M$$

whose g_j, $j\geq 1$, are defined by the morphisms

$$m^M_{i+2+j-1}(\mathbf{1}\otimes\omega^{\otimes i}\otimes\mathbf{1}^{\otimes j}) : M\otimes(SA)^{\otimes i}\otimes A\otimes A^{\otimes j-1}\to M, \quad i\geq 0, j\geq 1.$$

Let us show that the cone of the morphism

$$g_1 : M \overset{\infty}{\otimes}_A A \to M$$

is acyclic. We verify that the morphism of degree -1

$$r : M \otimes T^c SA \otimes SA \to M \otimes T^c SA \otimes SA$$

given by the morphism

$$\mathbf{1} \otimes s\eta : M \otimes (SA)^{\otimes i} \otimes SA \to M \otimes (SA)^{\otimes i+1} \otimes SA, \quad i \geq 0,$$

where η is the unit of A, is a contracting homotopy of the cone of g_1. □

Remark 4.1.1.7. The morphism of A-polydules g is clearly strictly unital. The morphism of A-polydules

$$f : M \to \overset{\infty}{\mathsf{Hom}}_A(A, M)$$

corresponding by adjunction to the morphism g is defined analogously to the morphism

$$f : A \to \mathsf{End} = \overset{\infty}{\mathsf{Hom}}_A(A, A)$$

of the key lemma (Lemma 5.3.0.1) of chapter 5. It is also strictly unital.

4.1.2 The derived category of an A_∞-algebra

Let $\mathbb{O}$ and $\mathbb{P}$ be two objects of C. Let A be an A_∞-algebra in $\mathsf{C}(\mathbb{O}, \mathbb{O})$. We recall that the category $\mathsf{Nod}_\infty\, A$ of A-polydules in $\mathsf{C}(\mathbb{P}, \mathbb{O})$ is isomorphic to the category $\mathsf{Mod}_\infty\, A^+$ of strictly unital A^+-polydules where A^+ is the augmentation of A. Consider the object $e = e_{\mathbb{O}}$ as an augmented A_∞-algebra in $\mathsf{C}(\mathbb{O}, \mathbb{O})$. Consider the object e as a strictly unital A^+-e-bipolydule thanks to the augmentation $A^+ \to e$. By section 4.1.1, we have a functor

$$? \overset{\infty}{\otimes}_{A^+} e : \mathsf{Mod}_\infty\, A^+ \to \mathsf{Mod}_\infty\, e.$$

It induces a functor in the derived categories which we denote in the same way.

Definition 4.1.2.1. The *derived category* of an A_∞-algebra is the kernel of the functor

$$? \overset{\infty}{\otimes}_{A^+} e : \mathcal{D}_\infty A^+ \to \mathcal{D}_\infty e.$$

Remark 4.1.2.2. We will show in (Remark 4.1.3.5) that a strictly unital A^+-polydule is in the kernel if and only if its bar construction is acyclic.

Remark 4.1.2.3. The theorem (4.1.3.1) below will show that this definition extends the definition of the derived category of an augmented A_∞-algebra (see Definition 2.4.2.1). In particular, we will show that if A is itself augmented, we have an exact sequence of triangulated categories

$$\mathcal{D}_\infty A \to \mathcal{D}_\infty A^+ \to \mathcal{D}_\infty e.$$

Theorem 4.1.2.4. Let A and A' be two A_∞-algebras and $f : A \to A'$ be an A_∞-quasi-isomorphism. The restriction along f induces an equivalence of categories

$$\mathcal{D}_\infty A' \to \mathcal{D}_\infty A.$$

Proof. Let $f^+ : A^+ \to A'^+$ be the augmented morphism associated to f. It is an A_∞-quasi-isomorphism. The functors

$$(\mathsf{Res}^{f^+}?) \overset{\infty}{\otimes}_{A^+} e \quad \text{and} \quad ? \overset{\infty}{\otimes}_{A'^+} e : \mathsf{Mod}_\infty A'^+ \to \mathsf{Mod}_\infty e$$

are therefore quasi-isomorphic. It therefore suffices to show that the restriction along f^+ induces an equivalence

$$\mathcal{D}_\infty A'^+ \to \mathcal{D}_\infty A^+.$$

The lemma (2.3.4.3) implies that the morphism between the enveloping algebras

$$U(f^+) : U(A^+) \to U(A'^+)$$

is a quasi-isomorphism. It follows [Kel94a, 6.1] that the restriction along $U(f^+)$ is an equivalence of categories

$$\mathcal{D}U(A'^+) \to \mathcal{D}U(A^+).$$

We deduce the result of the lemma (2.4.2.3). □

The case of H-unital A_∞-algebras

Definition 4.1.2.5. An A_∞-algebra is *H-unital* is an A_∞-algebra whose unaugmented bar construction is quasi-isomorphic to 0.

The notion of H-unital algebra is due to M. Wodzicki [Wod88]. It shows that an algebra is H-unital if and only if it satisfies the excision property (see [Wod88], [Wod89]).

Lemma 4.1.2.6. A minimal (i.e. $m_1 = 0$) strictly unital A_∞-algebra is H-unital.

Proof. Let (A, η) be a minimal strictly unital A_∞-algebra. The morphism of degree -1

$$h : BA \to BA$$

given by the morphisms

$$\mathbf{1} \otimes (s\eta) : (SA)^{\otimes i} \to (SA)^{\otimes i} \otimes SA$$

defines a contracting homotopy of BA. □

Corollary 4.1.2.7. A homologically unital A_∞-algebra (see the definition in section 3.1) is H-unital.

Proof. Let A be a homologically unital A_∞-algebra. The corollary (3.2.1.2) shows that A admits a strictly unital minimal model A'. Since BA' is weakly equivalent to BA and since weak equivalences are quasi-isomorphisms, we have the result. □

The subcategory $\mathsf{Tria}\, A$

Let $x : \mathbb{P} \to \mathbb{O}$ be a morphism of C. The morphism x induces a functor

$$x^* : \mathsf{C}(\mathbb{P}, \mathbb{O}) \to \mathsf{C}(\mathbb{P}, \mathbb{P}), \quad M \mapsto M(x).$$

We suppose that this functor admits a left adjoint

$$x_! : \mathsf{C}(\mathbb{P}, \mathbb{P}) \to \mathsf{C}(\mathbb{P}, \mathbb{O}).$$

Example 4.1.2.8. Let's look at the example appearing in the study of A_∞-categories (5.1.1). Let $\mathbb{P}$ and $\mathbb{O}$ be two sets and let

$$x : \mathbb{P} \to \mathbb{O}, \quad p \mapsto x(p)$$

be a map. The functor x^* sends $M \in C(\mathbb{P}, \mathbb{O})$ to

$$(p, p') \mapsto M(x(p), p').$$

The functor $x_!$ sends an object V of $\mathsf{C}(\mathbb{P}, \mathbb{P})$ to the $\mathbb{P}$-$\mathbb{O}$-bimodule

$$(o, p) \mapsto V(?, p) \otimes_{\mathbb{P}} e_{\mathbb{O}}(o, x(?)).$$

Now suppose that $\mathbb{P}$ is a one-element set. The map x is determined by the element $o = x(p)$ of $\mathbb{O}$. Let $V = e_{\mathbb{P}}$. The adjunction then gives us an isomorphism

$$\mathsf{Hom}_{\mathsf{C}(\mathbb{P},\mathbb{O})}(e_{\mathbb{O}}(?, o), M) \xrightarrow{\sim} M(o).$$

□

Let $x : \mathbb{P} \to \mathbb{O}$ be a morphism from C. Let V be an object of $\mathsf{C}(\mathbb{P}, \mathbb{P})$. Let's endow the object $x_!(V) \otimes_{\mathbb{O}} A$ with the structure of an A-polydule given by the multiplications of A. Since we have an isomorphism

$$\mathsf{Hom}_{\mathsf{C}(\mathbb{P},\mathbb{P})}(x_!(V), M) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Nod}_\infty A}(x_!(V) \otimes_{\mathbb{O}} A, M), \quad M \in \mathsf{Nod}_\infty A,$$

we have an adjunction

$$(x_!(?) \otimes_{\mathbb{O}} A, x^*) : \mathsf{C}(\mathbb{P}, \mathbb{P}) \to \mathsf{Nod}_\infty A.$$

Let $\mathsf{Tria}\, A$ be the smallest triangulated subcategory with infinite sums of $\mathcal{D}_\infty A^+$ containing the

$$x^\wedge = x_!(e_{\mathbb{P}}) \otimes_{\mathbb{O}} A, \quad x \in \mathsf{C}(\mathbb{P}, \mathbb{O}).$$

Remark 4.1.2.9. This notation is justified by the following fact. In the example appearing in the study of A_∞-categories (5.1.1), if $\mathbb{P}$ is a set of one element, and x is the map given by an element x of $\mathbb{O}$, the A-polydule $x^\wedge$ is the A_∞-function represented by x

$$x^\wedge = A(?, x).$$

Proposition 4.1.2.10. Let A be a H-unital A_∞-algebra. We have an exact sequence of triangulated categories

$$\mathsf{Tria}\, A \hookrightarrow \mathcal{D}_\infty A^+ \to \mathcal{D}_\infty e.$$

In particular, the derived category $\mathcal{D}_\infty A$ is equal to $\mathsf{Tria}\, A$.

In the case of differential graded algebras this proposition is proved in [Kel94b]. In the proof below, we use a filtration which is adapted from that of J. A. Guccione and J. J. Guccione [GG96]. It allows them to cleverly show the excision property of H-unital differential graded algebras.

Proof. Let us show that the composition

$$\mathsf{Tria}\, A \hookrightarrow \mathcal{D}_\infty A^+ \to \mathcal{D}_\infty e$$

is zero. As $x^\wedge$ is the A-polydule $x_!(e) \otimes_{\mathbb{O}} A$, it suffices to show that $A \overset{\infty}{\otimes}_{A^+} e$ is quasi-isomorphic to 0 in the category $\mathsf{C}(\mathbb{O}, \mathbb{O})$. We define a filtration of $A \overset{\infty}{\otimes}_{A^+} e = A \otimes T^c(SA^+) \otimes e$ by

$$F_p = \Big[\bigoplus_{0 \leq i < p} A \otimes (SA^+)^{\otimes i} \Big] \oplus \Big[\bigoplus_{0 \leq r} A \otimes (SA)^{\otimes r} \otimes (SA^+)^{\otimes p} \Big], \quad p \geq 0.$$

The F_p, $p \geq 0$, are subcomplexes of $A \overset{\infty}{\otimes}_{A^+} e$. The graded objects

$$\mathsf{Gr}_p A \overset{\infty}{\otimes}_{A^+} e = A \otimes T^c(SA^+) \otimes e = \bigoplus_{0 \leq r} A \otimes (SA)^{\otimes r} \otimes (Se)^{\otimes p}, \quad p \geq 0,$$

are isomorphic as complexes to

$$S^{-1} BA \otimes (Se)^{\otimes p}, \quad p \geq 0.$$

They are therefore acyclic, which shows that $A \overset{\infty}{\otimes}_{A^+} e$ is acyclic.

To prove that we have an exact sequence of triangulated categories, we are going to show that the inclusion of $\mathsf{Tria}\, A$ in $\mathcal{D}_\infty A^+$ has for right adjoint the functor

$$? \overset{\infty}{\otimes}_{A^+} A : \mathcal{D}_\infty A^+ \to \mathsf{Tria}\, A.$$

This amounts to showing that for each $X \in \mathsf{Mod}_\infty A^+$, the triangle

$$X \overset{\infty}{\otimes}_{A^+} A \to X \to X \overset{\infty}{\otimes}_{A^+} e \to S(X \overset{\infty}{\otimes}_{A^+} A)$$

is such that the object $X \overset{\infty}{\otimes}_{A^+} e \in \mathsf{Tria}\, e$ is $(\mathsf{Tria}\, A)$-local, i.e.

$$\mathsf{Hom}_{\mathcal{D}_\infty A^+}(L, X \overset{\infty}{\otimes}_{A^+} e) = 0, \quad L \in \mathsf{Tria}\, A.$$

As $A \overset{\infty}{\otimes}_{A^+} e$ is quasi-isomorphic to 0, the second arrow of the triangle of $\mathbb{O}$-$\mathbb{O}$-bimodules

$$A \overset{\infty}{\otimes}_{A^+} e \to A^+ \overset{\infty}{\otimes}_{A^+} e \to e \overset{\infty}{\otimes}_{A^+} e \to S(A \overset{\infty}{\otimes}_{A^+} e)$$

is an isomorphism in the derived category of A^+-polydules in $\mathsf{C}(\mathbb{O}, \mathbb{O})$. Moreover, the morphism

$$A^+ \overset{\infty}{\otimes}_{A^+} e \to e$$

is a quasi-isomorphism because its cone, which is the bar construction $BA^+ = T^c(SA^+)$, is acyclic (4.1.2.7). This implies that

$$e \to e \overset{\infty}{\otimes}_{A^+} e$$

is an isomorphism of A^+-A^+-bipolydules in $\mathsf{C}(\mathbb{O}, \mathbb{O})$. Let $X \in \mathcal{D}_\infty A^+$. Let us show that the object $X \overset{\infty}{\otimes}_{A^+} e \in \mathsf{Tria}\, e$ is $(\mathsf{Tria}\, A)$-local. Let L be an object of $\mathsf{Tria}\, A$ and a morphism

$$f : L \to X \overset{\infty}{\otimes}_{A^+} e.$$

We have a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & X \overset{\infty}{\otimes}_{A^+} e \arrow[d, "\sim"] \\ L \overset{\infty}{\otimes}_{A^+} e \arrow[r] & X \overset{\infty}{\otimes} e_{A^+} \overset{\infty}{\otimes}_{A^+} e, \end{tikzcd}$$

where the right vertical arrow represents an isomorphism of $\mathcal{D}_\infty A^+$ and where $L \overset{\infty}{\otimes}_{A^+} e$ is quasi-isomorphic to 0. The morphism f is therefore zero. □

4.1.3 The derived category of a strictly unital A_∞-algebra

Let A be a strictly unital A_∞-algebra. In this section, we give several descriptions of the derived category $\mathcal{D}_\infty A$ of (Definition 4.1.2.1). More precisely, we will show the following theorem:

Theorem 4.1.3.1. The following categories are equivalent:

D1. the derived category $\mathcal{D}_\infty A$ of (Definition 4.1.2.1), that is, the triangulated subcategory $\mathsf{Tria}\, A$ of $\mathcal{D}_\infty A^+$ (Proposition 4.1.2.10),

D2. the category (which we will show is well defined)

$$\mathcal{H}_\infty A : \mathsf{Mod}_\infty A/\sim$$

where $\sim$ is the homotopy relation (Definition 2.3.2.3),

D3. the localized category

$$\big(\mathsf{Mod}_\infty A\big)[Qis^{-1}]$$

where Qis is the class of A_∞-quasi-isomorphisms of $\mathsf{Mod}_\infty A$,

D4. the homotopy category

$$\mathsf{Ho}\,\mathsf{Mod}_\infty^{\mathsf{strict}} A$$

of the model category $\mathsf{Mod}_\infty^{\mathsf{strict}} A$ (defined below).

It follows from this theorem that if A is augmented, the definition of $\mathcal{D}_\infty A$ given in (Definition 2.4.2.1) is equivalent to that of (Definition 4.1.2.1).

Remark 4.1.3.2. The different descriptions of $\mathcal{D}_\infty A$ show that the results of Proposition (2.4.1.1) remain valid.

Equivalence between the categories of D1 and D2

As A is strictly unital, we have a strictly unital A_∞-morphism of A_∞-algebras

$$r = \begin{bmatrix} i \\ \eta \end{bmatrix} : A^+ = A \oplus e \to A$$

where η is the unit of A. We have a restriction functor

$$\mathsf{Res} : \mathsf{Mod}_\infty A \to \mathsf{Mod}_\infty A^+$$

which is faithful. We know that the isomorphism of categories (2.3.2)

$$\mathsf{Nod}_\infty A \xrightarrow{\sim} \mathsf{Mod}_\infty A^+$$

is compatible with homotopy. The proposition (3.3.1.8) shows that the restriction functor induces an isomorphism

$$\mathsf{Hom}_{\mathsf{Mod}_\infty A}(M, M')/\sim \longrightarrow \mathsf{Hom}_{\mathsf{Mod}_\infty A^+}(\mathsf{Res}\, M, \mathsf{Res}\, M')/\sim, \quad M, M' \in \mathsf{Mod}_\infty A,$$

where $\sim$ is the homotopy relation (2.3.2.3). The corollary (2.4.1.1) says that the homotopy relation (2.3.2.3) in $\mathsf{Mod}_\infty A^+$ is an equivalence relation compatible with composition. This shows that the homotopy relation in $\mathsf{Mod}_\infty A$ is an equivalence relation compatible with composition. So we have a well-defined category

$$\mathcal{H}_\infty A = \mathsf{Mod}_\infty A/\sim$$

and a fully faithful functor

$$J : \mathcal{H}_\infty A \hookrightarrow \mathcal{H}_\infty A^+ \simeq \mathcal{D}_\infty A^+.$$

Proposition 4.1.3.3. The restriction functor

$$\mathsf{Res} : \mathsf{Mod}_\infty A \to \mathsf{Mod}_\infty A^+$$

induces an equivalence of categories

$$\mathcal{H}_\infty A \to \mathsf{Tria}\, A.$$

Let's start by introducing a few notions.

Definition 4.1.3.4. An A^+-polydule is *H-unital* if its image under the functor

$$B : \mathsf{Mod}_\infty A^+ \to \mathsf{Comc}\, B^+ A^+$$

is quasi-isomorphic to 0.

Remark 4.1.3.5. An A^+-polydule M is H-unital if and only if the object $M \overset{\infty}{\otimes}_{A^+} e$ is quasi-isomorphic to 0. The sub-category of the H-unital A^+-polydules is therefore equal to the category $\mathsf{Tria}\, A$ by Proposition (4.1.2.10).

Remark 4.1.3.6. In the case where A is a unital associative algebra and M a unital module, the complex BM is the cone of the augmentation $\mathbf{p}M \to M$, where $\mathbf{p}M$ is the bar resolution of M (see for example [CE99, IX.6]). In particular, every unital A-module is a H-unital A^+-module.

Lemma 4.1.3.7. An A^+-polydule is H-unital if and only if it is homologically unital as an A-polydule.

Proof: Let M be a homologically unital A-polydule. There exists an A-polydule structure (necessarily homologically unital) on H^*M and an A_∞-quasi-isomorphism $H^*M \to M$. By the corollary 3.3.1.2, we can choose H^*M strictly unital. We then have a weak equivalence

$$B(H^*M) \to BM.$$

Since the weak equivalences are quasi-isomorphisms, it suffices to show that $B(H^*M)$ is quasi-isomorphic to 0. We verify that the morphism

$$r : SH^*M \otimes B^+(A^+) \to SH^*M \otimes B^+(A^+),$$

defined by morphisms

$$(\mathbf{I} \otimes s\eta) : SH^*M \otimes (SA)^{\otimes i} \to SH^*M \otimes (SA)^{\otimes i} \otimes SA, \quad i \geq 0,$$

where $\eta : e \to A$ is a strict unit of A, is a contracting homotopy of $B(H^*M)$.

To prove the converse we introduce some additional notions.

Generalized twisting cochains

Let C be an object of Cogca and A' an object of Alga_∞. A *generalized twisting cochain* $\tau : C \to A'$ is a graded morphism of degree $+1$ which vanishes on the co-augmentation ε^C, which is factorized by $\mathsf{ker}(A^+ \to e)$ and which satisfies

$$\sum_{i \geq 1} m_i \circ (\tau^{\otimes i}) \circ \Delta^{(i)} = 0.$$

Note that the infinite sum is well defined because τ vanishes on the co-augmentation and C is cocomplete.

Let M be an object of $\mathsf{Mod}_\infty A'$. We endow the tensor product $M \otimes C$ with the morphism of degree $+1$ which is the (well-defined) sum of the differential of the tensor product and the morphisms

$$M \otimes C \xrightarrow{\mathbf{1} \otimes \Delta^{(i)}} M \otimes C^{\otimes i} \xrightarrow{\mathbf{1} \otimes \tau^{\otimes i-1} \otimes \mathbf{1}} M \otimes A'^{\otimes i-1} \otimes C \xrightarrow{m_i \otimes \mathbf{1}} M \otimes C, \quad i \geq 1.$$

We verify that this morphism of degree $+1$ is a differential of $M \otimes C$. We denote by $M \otimes_\tau C$ the tensor product endowed with this differential. Let N be an object of $\mathsf{Comc}\, C$. We endow the tensor product $N \otimes A$ with the differential which is the (well-defined) sum of the differential of the tensor product and the morphisms

$$(\mathbf{1} \otimes m_i) \circ (\mathbf{1} \otimes \tau^{\otimes i-1} \otimes \mathbf{1}) \circ (\Delta^{(i)} \otimes \mathbf{1}) : N \otimes A' \to N \otimes A', \quad i \geq 1.$$

We equip $N \otimes A'$ with the morphism m_1 given by the above differential and with the morphisms m_i, $i \geq 2$, equal to $\mathbf{1}_N \otimes m_i^{A'}$. These morphisms define an A'-polydule structure on $N \otimes A'$. Let us denote this A'-polydule $N \otimes_\tau A'$. This gives us two functors

$$- \otimes_\tau A' : \mathsf{Comc}\, C \to \mathsf{Mod}_\infty A' \quad \text{and} \quad - \otimes_\tau C : \mathsf{Mod}_\infty A' \to \mathsf{Comc}\, C$$

called the *generalized twisted tensor product*.

End of the proof of Lemma 4.1.3.7. Let M be a H-unital A^+-polydule. We want to show that it is homologically unital as an A-polydule. We verify that the composition

$$B^+A^+ = T^cSA \xrightarrow{p_1} SA \xrightarrow{\omega} A \hookrightarrow A^+$$

is a generalized twisting element. We have a morphism of A^+-polydules

$$\eta^{B^+A^+} \otimes \varepsilon^{A^+} : B^+A^+ \otimes_\tau A^+ \to e$$

given by the unit of B^+A^+ and the co-augmentation of A^+. The morphism

$$M \otimes_\tau (\eta^{B^+A^+} \otimes \varepsilon^{A^+}) : M \otimes_\tau B^+A^+ \otimes_\tau A^+ \to M = M \otimes_\tau e$$

is a quasi-isomorphism (the contracting homotopy of the proof of the lemma 2.2.1.9 defines a contracting homotopy of its cone). The co-augmentation $A^+ \to e$ induces an exact sequence

$$0 \to M \otimes_\tau B^+A^+ \otimes_\tau A \xrightarrow{i} M \otimes_\tau B^+A^+ \otimes_\tau A^+ \to M \otimes_\tau B^+A^+ \otimes_\tau e \to 0.$$

The A^+-polydule M being H-unital, the object $M \otimes_\tau B^+A^+ \otimes_\tau e$ is quasi-isomorphic to 0 since it is isomorphic to $S^{-1}BM$. It follows that the morphism i is a quasi-isomorphism. The A^+-polydule M which is quasi-isomorphic to $M \otimes_\tau B^+A^+ \otimes_\tau A^+$ is thus quasi-isomorphic to $M \otimes_\tau B^+A^+ \otimes_\tau A$. As the latter is strictly unital over A, M is homologically unital over A. □

Proof of Proposition 4.1.3.3. We know that the functor

$$J : \mathcal{H}_\infty A \hookrightarrow \mathcal{H}_\infty A^+ \simeq \mathcal{D}_\infty A^+$$

is fully faithful. We must show that its image is made up of the objects of $\mathsf{Tria}\, A$. The lemma (4.1.3.7) shows that any object of $\mathsf{Mod}_\infty A$ is in $\mathsf{Tria}\, A$. Conversely, if an A^+-polydule M is in $\mathsf{Tria}\, A$, it is homologically unital over A. It is therefore (3.3.1.3) quasi-isomorphic to a strictly unital object. □

We endow the category $\mathcal{H}_\infty A$ with the triangulated structure induced by the equivalence

$$\mathcal{H}_\infty A \to \mathsf{Tria}\, A.$$

Equivalence between the categories of D2 and D3

The functor

$$J : \mathcal{H}_\infty A \to \mathcal{H}_\infty A^+$$

is fully faithful and we have an isomorphism of categories (Corollary 2.4.2.2)

$$\mathcal{H}_\infty A^+ \xrightarrow{\sim} \mathcal{D}_\infty A^+.$$

The A_∞-quasi-isomorphisms are therefore isomorphisms in $\mathcal{H}_\infty A$. As

$$\mathsf{Mod}_\infty A \to \mathcal{H}_\infty A$$

is a localization functor (with respect to homotopy equivalences), we have an isomorphism

$$\big(\mathsf{Mod}_\infty A\big)[Qis^{-1}] \xrightarrow{\sim} \mathcal{H}_\infty A.$$

Equivalence between the categories of D3 and D4

Let us start by showing some results on the derived category of a differential graded algebra.

Lemma 4.1.3.8. Let A be a unital differential graded algebra. The inclusion

$$J : \mathsf{Mod}\, A \to \mathsf{Mod}_\infty\, A$$

induces an equivalence

$$\mathcal{D}A \to \big(\mathsf{Mod}_\infty\, A\big)[Qis^{-1}].$$

Its inverse is given by the functor $? \overset{\infty}{\otimes}_A A$.

Proof. Consider A as an A-A-bipolydule. We associate to it (4.1.1.3) the functor

$$? \overset{\infty}{\otimes}_A A : \mathsf{Mod}_\infty\, A \to \mathsf{Mod}\, A.$$

We know by the lemma (4.1.1.6) that the A_∞-morphism

$$g_M : M \overset{\infty}{\otimes}_A A \to M, \quad M \in \mathsf{Mod}_\infty\, A,$$

is a A_∞-quasi-isomorphism. If M is a differential graded module over A, the multiplications m_i^M, $i \geq 3$, are zero and the A_∞-morphism g_M (constructed in the proof of lemma (4.1.1.6)) is strict. The A_∞-morphism g_M is then a morphism of differential graded A-modules. This shows that the functors J and $? \overset{\infty}{\otimes}_A A$ induce quasi-inverse functors of each other between categories

$$\mathcal{D}A \quad \text{and} \quad \big(\mathsf{Mod}_\infty\, A\big)[Qis^{-1}].$$

$\square$

Definition 4.1.3.9. Let A be a differential graded algebra (not necessarily unital). The *derived category* $\mathcal{D}A$ is the kernel of

$$? \overset{\mathbf{L}}{\otimes} e : \mathcal{D}A^+ \to \mathcal{D}e.$$

Remark 4.1.3.10. In the case where A is unital, the derived category defined above is equivalent to the derived category defined in (Section 2.2.3).

Corollary 4.1.3.11. Let A be a differential graded algebra (not necessarily unital). The derived categories $\mathcal{D}_\infty A$ and $\mathcal{D}A$ are equivalent.

Proof. This is a consequence of the lemma (4.1.3.8) and of the fact that the functor $? \overset{\mathbf{L}}{\otimes} e$ is exactly the functor $? \overset{\infty}{\otimes}_A e$. $\square$

The model category $\mathsf{Mod}_\infty^{\mathsf{strict}}\, A$

We use the standard differential graded operad notations and terminology below (see for example [Hin97]).

An *asymmetric operad* is a sequence of objects $\mathcal{O}(n)$, $n \geq 0$, of $\mathcal{C}\mathsf{C}$ endowed with a composition μ satisfying the same conditions of associativity of the composition of an operad in the usual sense. Let $\mathfrak{S}_n$, $n \geq 1$, denote the symmetric group. The sequence $\mathbb{K}\mathfrak{S}_n \otimes_{\mathbb{K}} \mathcal{O}(n)$, $n \geq 0$, is an S-module in $\mathcal{C}\mathsf{C}$ and μ induces a operad structure on this S-module. The operad Ass of associative algebras is equal to $\mathbb{K}\mathfrak{S}_n \otimes_{\mathbb{K}} \mathsf{Ass}'(n)$, $n \geq 0$, where Ass' is an asymmetric operad.

Let $\mathcal{O}$ be the *asymmetric operad of strictly unital* A_∞*-algebras*. We denote by $U(\mathcal{O}, A) = U(A)$ the enveloping algebra of A relative to the operad $\mathcal{O}$. The category $\mathsf{Mod}_\infty^{\mathsf{strict}} A$ of strictly unital A-polydules whose morphisms are the strict A_∞-morphisms is of course isomorphic to the category of (right) modules over the $\mathcal{O}$-algebra A. So we have an isomorphism of categories

$$\mathsf{Mod}\, U(A) \xrightarrow{\sim} \mathsf{Mod}_\infty^{\mathsf{strict}} A.$$

We deduce from Theorem (2.2.2.1) the following result.

Proposition 4.1.3.12. The three classes of morphisms below define a model category structure on $\mathsf{Mod}_\infty^{\mathsf{strict}} A$:

- the class $\mathcal{E}q$ of strict A_∞-quasi-isomorphisms,
- the class $\mathcal{F}ib$ of morphisms $f : M \to M'$ such that f^n is an epimorphism for all $n \in \mathbf{Z}$,
- the class $\mathcal{C}of$ of morphisms which have the left lifting property with respect to the morphisms belonging to $Qis \cap \mathcal{F}ib$.

□

We recall that the derived category $\mathcal{D}U(A)$ is isomorphic to the localized category

$$\mathsf{Ho}\left(\mathsf{Mod}_\infty^{\mathsf{strict}} A\right).$$

Remark 4.1.3.13. If A is an augmented A_∞-algebra, the enveloping algebra $U(A)$ is isomorphic to $\Omega^+ B^+ A$ (see 2.3.4.2).

Proposition 4.1.3.14. Let A be a strictly unital A_∞-algebra. The inclusion

$$J : \mathsf{Mod}_\infty^{\mathsf{strict}} A \to \mathsf{Mod}_\infty A$$

induces an equivalence

$$\mathsf{Ho}\left(\mathsf{Mod}_\infty^{\mathsf{strict}} A\right) \to \left(\mathsf{Mod}_\infty A\right)[Qis^{-1}].$$

Proof. First case: A is a unital differential graded algebra.
The sequence of inclusions

$$\mathsf{Mod}\, A \hookrightarrow \mathsf{Mod}_\infty^{\mathsf{strict}} A \hookrightarrow \mathsf{Mod}_\infty A$$

induces a sequence of faithful functors

$$\mathcal{D}A \to \mathsf{Ho}\left(\mathsf{Mod}_\infty^{\mathsf{strict}} A\right) \to \left(\mathsf{Mod}_\infty A\right)[Qis^{-1}].$$

The lemma (4.1.3.8) gives us the fully-faithful-ness of the composition. The second functor is therefore full and we have the result.

Second case: A is a strictly unital A_∞-algebra.

According to proposition (7.5.0.2), there exists a unital differential graded module A' and a trivial cofibration

$$i : A \to A'$$

that is strictly unital. The lemma (3.2.4.5) shows that there exists a trivial fibration $q : A' \to A$ such that $q \circ i = \mathbf{1}_A$ and $i \circ q$ is homotopic to $\mathbf{1}_{A'}$. The restriction functors Res^i and Res^q induce functors i^* and q^* between the homotopy categories

$$\mathsf{Ho}\left(\mathsf{Mod}_\infty^{\mathsf{strict}} A\right) \quad \text{and} \quad \mathsf{Ho}\left(\mathsf{Mod}_\infty^{\mathsf{strict}} A'\right).$$

We clearly have $i^* \circ q^* = \mathbf{1}$. Let us show that $q^* \circ i^*$ is isomorphic to the identity functor of $\mathsf{Ho}\left(\mathsf{Mod}_\infty^{\mathsf{strict}} A'\right)$.

Let A'^+ be the augmentation of A'. Its enveloping algebra $U(A'^+)$ is the differential graded algebra $\Omega^+ B^+ A'^+$ (see 2.3.4.4). Let $j : A'^+ \to U(A'^+)$ be the universal A_∞-morphism constructed in 2.3.4.3. Since it is an augmented strictly unital A_∞-quasi-isomorphism, it induces an equivalence

$$\mathcal{D}_\infty U(A'^+) \to \mathcal{D}_\infty A'^+$$

compatible with functors

$$\mathcal{D}_\infty A'^+ \to \mathcal{D}_\infty e \quad \text{and} \quad \mathcal{D}_\infty U(A'^+) \to \mathcal{D}_\infty e.$$

The subcategory $\mathcal{D}_\infty A' = \mathsf{Tria}\, A'$ is thus equivalent to the subcategory $\mathcal{D}_\infty \overline{U}(A'^+) = \mathsf{Tria}\, \overline{U}(A'^+)$ (the algebra $\overline{U}(A'^+) = \Omega B^+ A'^+$ is the reduction of $U(A'^+)$). Let f be the composed A_∞-morphism $i \circ q$. Let $f^+ \colon A'^+ \to A'^+$ be the augmented morphism associated with f. Let g be the morphism

$$\Omega^+ B^+ f^+ \colon U(A'^+) \to U(A'^+).$$

The morphism g is a morphism of unital differential graded algebras. To show that Res^f induces an endofunctor on $\mathsf{Ho}\,\mathsf{Mod}_\infty^{\mathsf{strict}} A'$ which is isomorphic to the identity functor, it suffices to show that Res^g induces an endofunctor on $\mathcal{D}U(A'^+)$ isomorphic to the identity functor. The morphism g is clearly homotopic to $\mathbf{1}$ in the category Alg_∞. The morphisms g and $\mathbf{1}$ therefore become equal in $\mathsf{Alg}[Qis^{-1}]$ (see 1.3.1.3). As $\Omega^+ B^+ A'^+$ is an almost free co-augmented algebra, it is a fibrant and cofibrant object of the model category Alg (see 1.3.1). There is therefore a right homotopy between $\mathbf{1}$ and g. The lemma (4.1.3.15) below shows that the endofunctor g^* of $\mathcal{D}\overline{U}(A'^+)$ induced by Res^g is isomorphic to identity. □

Lemma 4.1.3.15. Let A and B be two unital differential graded algebras. Let g and g' be two right-homotopic unital morphisms $A \to B$. The restriction functors along g and g' induce isomorphic functors

$$\mathcal{D}B \to \mathcal{D}A.$$

Proof. We recall that an algebra of paths B^I, that is to say an object of paths for B in the category of models Alg, is an object of Alg endowed with morphisms

$$B \xrightarrow{i} B^I \xrightarrow{p} B_0 \times B_1,$$

where B_0 and B_1 are equal to B, such that i is a weak equivalence and $p \circ i$ is a factorization of the diagonal $B \to B_0 \times B_1$. Let p_0 and p_1 be the composite morphisms

$$B^I \xrightarrow{p} B_0 \times B_1 \to B_0 \quad \text{and} \quad B^I \xrightarrow{p} B_0 \times B_1 \to B_1.$$

We have the equalities $p_0 \circ i = p_1 \circ i = \mathbf{1}$.

The morphisms g and g' are right homotopic with respect to the path algebra B^I, so there exists a morphism $H : A \to B^I$ such that $p_0 \circ H = g$ and $p_1 \circ H = g'$. This shows that

$$\mathsf{Res}^g = \mathsf{Res}^H \circ \mathsf{Res}^{p_0} \quad \text{and} \quad \mathsf{Res}^{g'} = \mathsf{Res}^H \circ \mathsf{Res}^{p_1} .$$

To show that Res^g and $\mathsf{Res}^{g'}$ induce isomorphic functors in derived categories, it suffices to show that Res^{p_0} and Res^{p_1} induce isomorphic functors in derived categories. We have equalities

$$\mathbf{1} = \mathsf{Res}^i \circ \mathsf{Res}^{p_0} = \mathsf{Res}^i \circ \mathsf{Res}^{p_1} .$$

As i is a quasi-isomorphism, Res^i induces an equivalence in the derived categories. We deduce that Res^{p_0} and Res^{p_1} induce isomorphic functors in derived categories. □

4.2 The derived category of A-infinity bimodules

Proofs in this section are omitted because they are similar to those in section 4.1.

The functor $M \overset{\infty}{\otimes} ? \overset{\infty}{\otimes} M''$

Let $\mathbb{O}$, $\mathbb{O}'$, $\mathbb{O}''$ and $\mathbb{O}'''$ be objects of C. Let A (resp. A', A'', A''') be an A_∞-algebra in $\mathsf{C}(\mathbb{O},\mathbb{O})$ (resp. $\mathsf{C}(\mathbb{O}',\mathbb{O}')$, $\mathsf{C}(\mathbb{O}'',\mathbb{O}'')$, $\mathsf{C}(\mathbb{O}''',\mathbb{O}''')$). Let M (resp. M'') be a A-A'-bipolydule (resp. A''-A'''-bipolydule) in $\mathsf{C}(\mathbb{O},\mathbb{O}')$ (resp. $\mathsf{C}(\mathbb{O}'',\mathbb{O}''')$). We define the functor

$$\mathsf{Nod}_\infty(A', A'') \to \mathsf{Nod}_\infty(A, A'''), \quad M' \mapsto M \overset{\infty}{\otimes} M' \overset{\infty}{\otimes} M,$$

by the equality of differential graded B^+A-B^+A'''-bicomodules

$$B(M \overset{\infty}{\otimes} M' \overset{\infty}{\otimes} M) = BM \,\square_{B^+A'} BM' \,\square_{B^+A''} BM,$$

where $\square$ designates the cotensor product (see 4.1.1).

The derived category $\mathcal{D}_\infty(A, A')$

Let $e_{\mathbb{O}}$ and $e_{\mathbb{O}'}$ be the neutral elements of $\mathsf{C}(\mathbb{O},\mathbb{O})$ and $\mathsf{C}(\mathbb{O}',\mathbb{O}')$ considered as augmented A_∞-algebras. Consider $e_{\mathbb{O}}$ and $e_{\mathbb{O}'}$ as a $e_{\mathbb{O}}$-A^+-bipolydule and an A'^+-$e_{\mathbb{O}'}$-bipolydule.

Definition 4.2.0.1. The *derived category* $\mathcal{D}_\infty(A', A'')$ is the kernel of the functor

$$e_{\mathbb{O}} \overset{\infty}{\otimes}_{A'^+} ? \overset{\infty}{\otimes}_{A''^+} e_{\mathbb{O}'} : \mathcal{D}_\infty(A'^+, A''^+) \to \mathcal{D}_\infty(e_{\mathbb{O}}, e_{\mathbb{O}'}).$$

The subcategory $\mathsf{Tria}(A, A')$

Suppose that the category C admits a final object $\mathbb{P}$. Let $x : \mathbb{P} \to \mathbb{O}$ be a morphism from C. The morphism x induces a functor

$$x_* : \mathsf{C}(\mathbb{O},\mathbb{P}) \to \mathsf{C}(\mathbb{P},\mathbb{P}), \quad M \mapsto M(x).$$

We assume that this functor admits a left adjoint

$$_!x : \mathsf{C}(\mathbb{P},\mathbb{P}) \to \mathsf{C}(\mathbb{P},\mathbb{O}).$$

We have a left A-polydule

$$x^\vee = A \otimes_{\mathbb{O}} {}_!x(e_{\mathbb{P}}),$$

whose structure is given by the multiplications of A.

Remark 4.2.0.2. This notation is justified by the following fact. In the example appearing in the study of A_∞-categories (5.1.1), a final object is a one-element set. Let $\mathbb{P}$ be such a set and $\mathbb{O}$ a set. Let x be the map $\mathbb{P} \to \mathbb{O}$ given by an element (also denoted x) of $\mathbb{O}$. The A-polydule $x^\vee$ is the A_∞-functor co-represented by x

$$x^\vee = A(x, ?).$$

$\square$

Let $x : \mathbb{P} \to \mathbb{O}$ and $y : \mathbb{P} \to \mathbb{O}'$ be morphisms of C. The $\mathbb{O}$-$\mathbb{O}'$-bimodule

$$x^\vee \otimes_{\mathbb{P}} y^\wedge = A \otimes_{\mathbb{O}'} {}_!x(e_{\mathbb{P}}) \otimes_{\mathbb{P}} y_!(e_{\mathbb{P}}) \otimes_{\mathbb{O}'} A'$$

is an A-A'-bipolydule. The category $\mathsf{Tria}(A, A')$ is the triangulated subcategory of $\mathcal{D}_\infty(A^+, A'^+)$ generated by

$$x^\vee \otimes_{\mathbb{P}} y^\wedge, \quad x \in \mathsf{C}(\mathbb{P},\mathbb{O}), \quad y \in \mathsf{C}(\mathbb{P},\mathbb{O}').$$

Proposition 4.2.0.3. Let A and A' be two H-unital A_∞-algebras. We have a exact sequence of triangulated categories

$$\mathsf{Tria}(A, A') \hookrightarrow \mathcal{D}_\infty(A^+, A'^+) \to \mathcal{D}_\infty(e_{\mathbb{O}}, e'_{\mathbb{O}}).$$

In particular, the derived category $\mathcal{D}_\infty A$ is equal to $\mathsf{Tria}(A, A')$. $\square$

Theorem 4.2.0.4. Let A and A' be two strictly unital A_∞-algebras. The following categories are equivalent:

D1. the derived category $\mathcal{D}_\infty(A, A')$ of (Definition 4.2.0.1), that is, the triangulated subcategory $\mathsf{Tria}(A, A')$ of $\mathcal{D}_\infty(A^+, A'^+)$ (Proposition 4.1.2.10),

D2. the category (well defined)

$$\mathcal{H}_\infty(A, A') = \mathsf{Mod}_\infty(A, A')/\sim$$

where $\sim$ is the homotopy relation,

D3. the localized category

$$\big(\mathsf{Mod}_\infty(A, A')\big)[Qis^{-1}]$$

where Qis is the class of A_∞-quasi-isomorphisms of $\mathsf{Mod}_\infty(A, A')$,

D4. the localized category

$$\big(\mathsf{Mod}_\infty^{\mathsf{strict}}(A, A')\big)[Qis^{-1}]$$

of the category $\mathsf{Mod}_\infty^{\mathsf{strict}}(A, A')$.

Proof. The equivalences between the categories of D1, D2 and D3 are shown in the same way as in the theorem (4.1.3.1). The equivalence between the categories of D3 and D4 in the case where A and A' are unital differential graded algebras is proved as in Proposition (4.1.3.14). If A and A' are any strictly unital A_∞-algebras, we proceed as follows. We show as in Proposition (4.1.3.14) that the inclusion

$$\mathsf{Mod}(U(A), U(A')) \hookrightarrow \mathsf{Mod}_\infty(A, A')$$

induces an equivalence

$$\mathsf{Ho}\left(\mathsf{Mod}(U(A), U(A'))\right) \to \left(\mathsf{Mod}_\infty(A, A')\right)[Qis^{-1}].$$

As this equivalence is the composition of faithful functors

$$\mathsf{Ho}\left(\mathsf{Mod}(U(A), U(A'))\right) \to \left(\mathsf{Mod}_\infty^{\mathsf{strict}}(A, A')\right)[Qis^{-1}] \xrightarrow{K} \left(\mathsf{Mod}_\infty(A, A')\right)[Qis^{-1}]$$

the functor K is full. It is therefore an equivalence. $\square$

Chapter 5

A_∞-categories and A_∞-functors

Chapter plan

An A_∞-category is an A_∞-algebra with several objects, and conversely, an A_∞-algebra is an A_∞-category with one object. The problems raised by the increase in the number of objects are numerous and the generalization of the results of the previous chapters is sometimes very technical.

In the section 5.1, we fix notations which encode the variation of the sets of objects of the small A_∞-categories.

For this, we introduce a bicategory C whose objects are the sets, then we define a small A_∞-category whose set of objects is in bijection with a set $\mathbb{O}$ as an A_∞-algebra in the (monoidal) category $\mathsf{C}(\mathbb{O},\mathbb{O})$. We then define the A_∞-functors.

In the section 5.2, we define the differential graded categories of (bi)polydules over A_∞-categories.

In the section 5.3, we establish a lemma (called the *key lemma*) which will be fundamental in the construction of the Yoneda A_∞-functor (Definition 7.1.0.1) and of the generalized Yoneda A_∞-functor (Section 8.2.1).

5.1 Definitions

5.1.1 The base categories $\mathsf{C}(\mathbb{O},\mathbb{O}')$ and $\mathsf{C}(\mathbb{O})$

We fix notations that we will use throughout this part. We construct a bicategory C whose objects are the sets (see [ML98, Chap. XII, §6] for the bicategories).

Let $\mathbb{K}$ be a field. The tensor product above $\mathbb{K}$ is denoted $\otimes$. Let $\mathbb{O}$ be a set. Consider it as the *small category* whose objects are in bijection with $\mathbb{O}$ and whose space of morphisms $o \to o'$ is empty if $o \neq o'$, and contains only the identity morphism $\mathbf{I}_o$ otherwise.

Let $\mathbb{O}$, $\mathbb{O}'$ and $\mathbb{O}''$ be three sets. A $\mathbb{O}'$-$\mathbb{O}$-*bimodule* (resp. a right $\mathbb{O}$-*module*) is a functor

$$M : \mathbb{O}^{op} \times \mathbb{O}' \to \mathsf{Vect}\mathbb{K}, \quad \Big(\text{resp.} \quad M : \mathbb{O}^{op} \to \mathsf{Vect}\mathbb{K}\Big)$$

where $\mathsf{Vect}\mathbb{K}$ is the category of $\mathbb{K}$-vector spaces. A *morphism* of bimodules (resp. of modules) is a morphism of functors. We denote these categories by $\mathsf{C}(\mathbb{O},\mathbb{O}')$ and $\mathsf{C}(\mathbb{O})$. Let M be an object of $\mathsf{C}(\mathbb{O},\mathbb{O}')$ and N an object of $\mathsf{C}(\mathbb{O}',\mathbb{O}'')$. The *tensor product* $M\odot_{\mathbb{O}'}N$ *above* $\mathbb{O}'$ is the object of $\mathsf{C}(\mathbb{O},\mathbb{O}'')$ defined by

$$\Big(M\odot_{\mathbb{O}'}N\Big)(o'',o)=\bigoplus_{o'\in\mathbb{O}'}M(o',o)\otimes N(o'',o').$$

We'll simply denote the tensor product above $\mathbb{O}'$ by $\odot$ when it won't cause confusion. The tensor product above $\mathbb{O}'$ gives us a functor

$$\mathsf{C}(\mathbb{O}',\mathbb{O})\times\mathsf{C}(\mathbb{O}'',\mathbb{O}')\to\mathsf{C}(\mathbb{O}'',\mathbb{O}'),\quad (M,N)\mapsto M\odot_{\mathbb{O}'}N,$$

and if $\mathbb{O}'''$ is a set and T an object of $\mathsf{C}(\mathbb{O}'',\mathbb{O}''')$, we have associativity constraints

$$(M\odot_{\mathbb{O}'}N)\odot_{\mathbb{O}''}T\xrightarrow{\sim}M\odot_{\mathbb{O}'}(N\odot_{\mathbb{O}''}T).$$

Let $f:\mathbb{O}\to\mathbb{O}'$ be a map. We have a functor

$$\mathsf{C}(\mathbb{O}'',\mathbb{O}')\longrightarrow\mathsf{C}(\mathbb{O}'',\mathbb{O}),$$

which sends the $\mathbb{O}'$-$\mathbb{O}''$-bimodule M to the $\mathbb{O}$-$\mathbb{O}''$-bimodule

$$\begin{array}{rcl} M_f:\mathbb{O}^{op}\times\mathbb{O}'' & \to & \mathsf{Vect}\mathbb{K}\\ o\times o'' & \mapsto & M(fo,o''). \end{array}$$

Similarly, if $g:\mathbb{O}\to\mathbb{O}''$ is a map, we have a functor

$$\mathsf{C}(\mathbb{O}'',\mathbb{O}')\longrightarrow\mathsf{C}(\mathbb{O},\mathbb{O}'),\quad M\mapsto {}_gM.$$

The category $\mathsf{C}(\mathbb{O},\mathbb{O}')$ is $\mathbb{K}$-linear, abelian, semi-simple, cocomplete, with exact filtrant colimits (i.e. it is a semi-simple Grothendieck $\mathbb{K}$-category). By the section 1.1.1, we have the categories $\mathcal{G}r\mathsf{C}(\mathbb{O},\mathbb{O}')$ of *graded bimodules* and $\mathcal{C}\mathsf{C}(\mathbb{O},\mathbb{O}')$ of *differential graded bimodules*. Note that the tensor product $\odot_{\mathbb{O}}$ and the bimodule

$$e_{\mathbb{O}}(-,-)=\mathbb{K}\mathsf{Hom}_{\mathbb{O}}(-,-)$$

define a monoidal category structure on $(\mathsf{C}(\mathbb{O},\mathbb{O}),\odot,e_{\mathbb{O}})$. The functor

$$\mathsf{C}(\mathbb{O}',\mathbb{O}')\to\mathsf{C}(\mathbb{O},\mathbb{O}),\quad M\mapsto {}_fM_f,$$

is compatible with the monoidal structure. Since the category $\mathsf{C}(\mathbb{O})$ is isomorphic to $\mathsf{C}(\{*\},\mathbb{O})$, where $\{*\}$ is a singleton set, we get a right action on the monoidal category $C(\mathbb{O},\mathbb{O})$ over $C(\mathbb{O})$

$$\mathsf{C}(\mathbb{O})\times\mathsf{C}(\mathbb{O},\mathbb{O})\to\mathsf{C}(\mathbb{O}),\quad (M,N)\mapsto M\odot N.$$

Remark 5.1.1.1. Let $\mathcal{A}$ be a small $\mathbb{K}$-category whose set of objects is in bijection with a set $\mathbb{A}$. The $\mathbb{A}$-$\mathbb{A}$-bimodule

$$\mathsf{Hom}_{\mathcal{A}}:A\times A'\mapsto\mathsf{Hom}_{\mathcal{A}}(A,A'),$$

equipped with morphisms

$$\mu:\mathsf{Hom}_{\mathcal{A}}\odot\mathsf{Hom}_{\mathcal{A}}\to\mathsf{Hom}_{\mathcal{A}}\quad\text{and}\quad\eta:e_{\mathbb{A}}\to\mathsf{Hom}_{\mathcal{A}},\quad\mathbf{I}_A\mapsto\mathbf{1}_A,$$

given by the composition of $\mathcal{A}$ and by the identity morphisms $\mathbf{1}_A$ of $\mathcal{A}$, is a unital algebra in the category of $\mathbb{A}$-$\mathbb{A}$-bimodules. Conversely, a unital algebra in the category of $\mathbb{A}$-$\mathbb{A}$-bimodules defines a small $\mathbb{K}$-category whose set of objects is in bijection with $\mathbb{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two small $\mathbb{K}$-categories whose sets of objects are in bijection with the sets $\mathbb{A}$ and $\mathbb{B}$. Let $f : \mathcal{A} \to \mathcal{B}$ be a functor. We notice

$$\dot{f} : \mathsf{Obj}\,\mathcal{A} \to \mathsf{Obj}\,\mathcal{B}$$

the map which sends A on its image by the functor f. The functor f induces a morphism of unital algebras

$$\mathsf{Hom}_{\mathcal{A}} \to {}_{\dot{f}}\mathsf{Hom}_{\mathcal{B}\,\dot{f}}, \quad x \mapsto f(x),$$

Conversely, if Λ and Λ' are two unital algebras in the categories of $\mathbb{A}$-$\mathbb{A}$-bimodules and $\mathbb{B}$-$\mathbb{B}$-bimodules, a map $\dot{f} : \mathbb{A} \to \mathbb{B}$ and a unital algebra morphism $\Lambda \to {}_{\dot{f}}\Lambda'_{\dot{f}}$ in the category $\mathsf{C}(\mathbb{A}, \mathbb{A})$ define a functor between the corresponding $\mathbb{K}$-categories.

Definition 5.1.1.2. Let $\mathbb{A}$ be a set. A *(small) dg category over* $\mathbb{A}$ is a unital dg algebra in $\mathsf{C}(\mathbb{A}, \mathbb{A})$.

5.1.2 Definitions

Definition 5.1.2.1. Let $\mathbb{A}$ be a set. An A_∞*-category over* $\mathbb{A}$ is an A_∞-algebra in the category

$$(\mathcal{G}r\mathsf{C}(\mathbb{A}, \mathbb{A}), \odot, e_{\mathbb{A}}).$$

Remark 5.1.2.2. Let $\mathcal{A}$ be an A_∞-category. It is determined by

- a set of *objects* $\mathsf{Obj}\,\mathcal{A} = \mathbb{A}$,
- for all pairs (A, A') of objects of $\mathcal{A}$, a graded vector space of *morphisms*

$$\mathsf{Hom}_{\mathcal{A}}(A, A') = \mathcal{A}(A, A'),$$

- for all sets $(A_0, \ldots, A_n)$ of objects of $\mathcal{A}$, the *compositions*

$$m_n : \mathcal{A}(A_{n-1}, A_n) \otimes \ldots \otimes \mathcal{A}(A_0, A_1) \to \mathcal{A}(A_0, A_n),$$

  satisfying the equations $(*_n)$, $n \geq 1$, of the definition 1.2.1.1,

If $\mathcal{A}$ is homologically unital (as A_∞-algebra in $\mathcal{G}r\mathsf{C}(\mathbb{A}, \mathbb{A})$) then, for any object $A \in \mathcal{A}$, we have an *identity morphism* $\mathbf{I}_A \in \mathcal{A}(A, A)$ such that its class $[\mathbf{I}_A]$ in $H^*\mathcal{A}(A, A)$ satisfies

$$\mu(f, [\mathbf{I}_A]) = f, \quad f \in H^*\mathcal{A}(A, A') \quad \text{and} \quad \mu([\mathbf{I}_A], g) = g, \quad g \in H^*\mathcal{A}(A', A),$$

where μ is the composition of $H^*\mathcal{A}$ induced by m_2.

Remark 5.1.2.3. The composition m_2 induces an associative composition

$$\mu : H^0\mathcal{A} \odot H^0\mathcal{A} \to H^0\mathcal{A}.$$

If $\mathcal{A}$ is homologically unital then $H^0\mathcal{A}$ is a category in the classical sense. The identity morphism of an object $A \in H^0\mathcal{A}$ is the class $[\mathbf{I}_A]$.

Lemma 5.1.2.4. Let $\mathbb{B}$ be a set, $\mathcal{B}$ a homologically unital A_∞-category over $\mathbb{B}$ and

$$f : \mathbb{A} \to \mathbb{B}$$

a map. The graded $\mathbb{A}$-$\mathbb{A}$-bimodule ${}_f\mathcal{B}_f$ is a homologically unital A_∞-category with compositions and identity morphisms induced by those of $\mathcal{A}$. □

Definition 5.1.2.5. Let $\mathbb{A}$ and $\mathbb{B}$ be two sets and $\mathcal{A}$ and $\mathcal{B}$ two A$_\infty$-catgories over $\mathbb{A}$ and $\mathbb{B}$. An A$_\infty$-*functor*

$$f : \mathcal{A} \to \mathcal{B}$$

is the data of a pair $(\dot{f}, f_{\mathsf{Hom}})$ consisting of a map

$$\dot{f} : \mathbb{A} \to \mathbb{B}$$

and an A$_\infty$-morphism in the category $\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})$

$$f_{\mathsf{Hom}} : \mathcal{A} \to {}_{\dot{f}}\mathcal{B}_{\dot{f}}.$$

We will often denote the latter by f instead of f_{Hom}. The A$_\infty$-functor *identity of* $\mathcal{A}$ is denoted

$$\mathbf{1}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}.$$

Be careful not to confuse this symbol with $\mathbf{I}_A$, the identity morphism of an object $A \in \mathcal{A}$.

Remark 5.1.2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two small A$_\infty$-categories. An A$_\infty$-function $f : \mathcal{A} \to \mathcal{B}$ is determined by

- a map
$$\dot{f} : \mathsf{Obj}\,\mathcal{A} \to \mathsf{Obj}\,\mathcal{B},$$
- for any sequence $(A_0, \ldots, A_n)$ of objects of $\mathcal{A}$, morphisms
$$f_n : \mathcal{A}(A_{n-1}, A_n) \otimes \ldots \otimes \mathcal{A}(A_0, A_1) \to \mathcal{B}(\dot{f}A_0, \dot{f}A_n),$$
satisfying the equations $(**_n)$, $n \geq 1$, of definition 1.2.1.2.

Remark 5.1.2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be two small A$_\infty$-categories over $\mathbb{A}$. An A$_\infty$-morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathsf{C}(\mathbb{A},\mathbb{A})$ yields an A$_\infty$-function

$$(\mathbf{1}_{\mathbb{A}}, f) : \mathcal{A} \to \mathcal{B}, \quad x \mapsto f(x).$$

Conversely, an A$_\infty$-function $(\dot{f}, f)$ whose underlying map $\dot{f}$ is equal to $\mathbf{1}_{\mathbb{A}}$ yields an A$_\infty$-morphism $f : \mathcal{A} \to \mathcal{B}$.

Recalling the bar construction

Let $\mathcal{A}$ and $\mathcal{B}$ be two A$_\infty$-categories and $f : \mathcal{A} \to \mathcal{B}$ an A$_\infty$-functor. Recall that the bijections in section 1.2.2,

$$m_i \leftrightarrow b_i \quad \Big(\text{resp.} \quad f_i \leftrightarrow F_i\Big), \quad i \geq 1,$$

between the morphism spaces

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(\mathcal{A}^{\odot i}, \mathcal{A}\big) \quad \text{and} \quad \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big((S\mathcal{A})^{\odot i}, S\mathcal{A}\big)$$

$$\Big(\text{resp.} \quad \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(\mathcal{A}^{\odot i}, {}_{\dot{f}}\mathcal{B}_{\dot{f}}\big) \quad \text{and} \quad \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big((S\mathcal{A})^{\odot i}, {}_{\dot{f}}\mathcal{B}_{\dot{f}}\big)\Big)$$

are defined by the relations

$$\omega \circ b_i = -m_i \circ \omega^{\odot i} \quad \Big(\text{resp.} \quad \omega \circ F_i = (-1)^{|F_i|} f_i \circ \omega^{\odot j}\Big),$$

where F_i is a graded morphism of degree $|F_i|$ and $\omega = s^{-1}$. A functor $f : \mathcal{A} \to \mathcal{B}$ is the data of a map $\dot{f} : \mathsf{Obj}\,\mathcal{A} \to \mathsf{Obj}\,\mathcal{B}$ and a differential graded morphism

$$F = Bf : B\mathcal{A} \to B_{\dot{f}}\mathcal{B}_{\dot{f}}$$

in the category $\mathsf{Cogc}(\mathbb{A}, \mathbb{A})$ of differential graded cocomplete coalgebras of $\mathsf{C}(\mathbb{A}, \mathbb{A})$.

Definition 5.1.2.8. Let $\mathbb{A}$ be a set and $\mathcal{A}$ an A_∞-category over $\mathbb{A}$. an *$\mathcal{A}$-polydule* is an $\mathcal{A}$-polydule over $\mathcal{G}r\mathsf{C}(\mathbb{A})$ (see Definition 2.3.1.2). It is given by a right $\mathbb{A}$-module

$$M : \mathbb{A}^{op} \to \mathcal{G}r\mathsf{C}$$

endowed with graded morphisms of right $\mathbb{A}$-modules

$$m_i : M \odot \mathcal{A}^{\odot i-1} \to M, \quad i \geq 1,$$

of degree $2-i$, such that an equation $(*'_n)$, $n \geq 1$, of the same form as the equation $(*_n)$, $n \geq 1$, of Definition 1.2.1.1 is satisfied.

Remark 5.1.2.9. Let V be an object of $\mathsf{C}(\mathbb{A})$. The $\mathbb{A}$-module $V \odot \mathcal{A}$ equipped with the morphisms

$$\mathbf{1}_V \otimes m_i : \big(V \odot \mathcal{A}\big) \odot \mathcal{A}^{\odot i-1} \to V \odot \mathcal{A}, \quad i \geq 1,$$

is an $\mathcal{A}$-polydule. In particular, if A is an object of $\mathcal{A}$, the $\mathbb{A}$-module

$$\mathcal{A}(-, A) = e(-, A) \odot \mathcal{A}$$

is an $\mathcal{A}$-polydule in $\mathsf{C}(\mathbb{A})$. We will denote it $A^\wedge$.

Definition 5.1.2.10. Let $\mathbb{A}$ and $\mathbb{B}$ be two sets and $\mathcal{A}$ and $\mathcal{B}$ two A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$. A *$\mathcal{A}$-$\mathcal{B}$-bipolydule* is a $\mathcal{A}$-$\mathcal{B}$-bipolydule in $\mathcal{G}r\mathsf{C}(\mathbb{A}, \mathbb{B})$ (see Definition 2.5.1.3).

5.2 Differential graded categories of polydules

The category $\mathcal{C}_\infty B^+\mathcal{A}$

Let $\mathbb{A}$ be a set and $\mathcal{A}$ an A_∞-category over $\mathbb{A}$. The *category* $\mathcal{C}_\infty B^+\mathcal{A}$ has as its objects those of $\mathsf{Comc}\,B^+\mathcal{A}$. If N and N' are two objects of $\mathsf{Comc}\,B^+\mathcal{A}$, the space of morphisms

$$\mathsf{Hom}_{\mathcal{C}_\infty B^+\mathcal{A}}(N, N')$$

is the space of graded unital morphisms of comodules $N \to N'$ endowed with the differential

$$\delta : F \mapsto b^{N'} \circ F - (-1)^{|F|} F \circ b^N,$$

where F has degree $|F|$. It is a differential graded category. Note that the category $\mathsf{Comc}\,B^+\mathcal{A}$ is isomorphic to the category $Z^0\mathcal{C}_\infty B^+\mathcal{A}$, i.e. the category whose objects are those of $\mathcal{C}_\infty B^+\mathcal{A}$ and whose morphisms are the zero-cycles of the morphism complexes of $\mathcal{C}_\infty B^+\mathcal{A}$.

The category $\mathcal{N}_\infty\mathcal{A}$

The *category* $\mathcal{N}_\infty\mathcal{A}$ is the dg category whose objects are $\mathcal{A}$-polydules and whose morphism spaces are defined by

$$\mathsf{Hom}_{\mathcal{N}_\infty\mathcal{A}}(M,M') = \mathsf{Hom}_{\mathcal{C}_\infty B^+\mathcal{A}}(BM,BM'), \quad M,M' \in \mathcal{N}_\infty\mathcal{A}.$$

A morphism $f : M \to M'$ of degree n is thus given by a sequence of graded morphisms of $\mathbb{A}$-modules

$$f_i : M \odot \mathcal{A}^{\odot i-1} \to M'$$

of degree $1-i+n$. The A_∞-morphisms $f : M \to M'$ are the zero-cycles of $\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(M,M')$. (The letter $\mathcal{N}$ refers to the "$\mathcal{N}$on" in "$\mathcal{A}$-nonunital polydules".)

Remark 5.2.0.1. Let $\mathcal{B}$ be an A_∞-category and X a $\mathcal{B}$-$\mathcal{A}$-bipolydule. We have an isomorphism of complexes

$$\mathsf{H\overset{\infty}{o}m}_{\mathcal{A}}(X,M) = \mathsf{Hom}_{\mathcal{N}_\infty\mathcal{A}}(X_{\mathcal{A}},M), \quad M \in \mathcal{N}_\infty\mathcal{A},$$

where $\mathsf{H\overset{\infty}{o}m}_{\mathcal{A}}(X,M)$ is defined in (Section 4.1.1).

The category $\mathcal{C}_\infty\mathcal{A}$

Suppose now that $\mathcal{A}$ is strictly unital. If M and M' are two strictly unital $\mathcal{A}$-polydules, a morphism $f : M \to M'$ of degree n is *strictly unital* if it satisfies the equations

$$f_i(\mathbf{1}^{\otimes\alpha} \otimes \eta \otimes \mathbf{1}^{\otimes\beta}) = 0, \quad i \geq 2.$$

We denote by $\big(\mathcal{N}_\infty\mathcal{A}\big)_u$ the full *subcategory* of $\mathcal{N}_\infty\mathcal{A}$ formed from strictly unital $\mathcal{A}$-polydules and $\mathcal{C}_\infty\mathcal{A}$ the non-full *subcategory* of $\mathcal{N}_\infty\mathcal{A}$ formed of the strictly unital $\mathcal{A}$-polydules whose morphisms are the strictly unital morphisms. Note that if $\mathcal{A}$ is augmented, we have an isomorphism of categories

$$\mathcal{C}_\infty\mathcal{A} \xrightarrow{\sim} \mathcal{N}_\infty\overline{\mathcal{A}}.$$

Remark 5.2.0.2. The category $H^0\mathcal{C}_\infty\mathcal{A}$ is clearly isomorphic to $\mathcal{H}_\infty\mathcal{A}$ (see the definition 4.1.2.1). It is equivalent to the category $\mathcal{D}_\infty\mathcal{A}$ by Corollary 2.4.2.2.

Proposition 5.2.0.3. The inclusion

$$\mathcal{C}_\infty\mathcal{A} \to \mathcal{N}_\infty\mathcal{A}$$

induces a quasi-isomorphism on the morphism spaces.

Proof. The proof is the same as that of the proposition (3.3.1.8). Instead of considering only the A_∞-morphisms, i.e. the morphisms of $\mathcal{N}_\infty\mathcal{A}$ which are cycles of degree zero and the homotopies between A_∞-morphisms, we consider morphisms which are cycles of any degree. □

The category $\mathcal{C}_\infty(\mathcal{A},\mathcal{B})$

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets and $\mathcal{A}$ and $\mathcal{B}$ be A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$. The category $\mathcal{N}_\infty(\mathcal{A},\mathcal{B})$ is constructed in a strictly analogous way to $\mathcal{N}_\infty\mathcal{A}$. Let $\mathcal{C}_\infty(B^+\mathcal{A}, B^+\mathcal{B})$ be the the dg category whose objects are those of $\mathsf{Comc}(B^+\mathcal{A}, B^+\mathcal{B})$. The *category* $\mathcal{N}_\infty(\mathcal{A},\mathcal{B})$ is the dg category whose objects are the same as those of $\mathsf{Nod}_\infty(\mathcal{A},\mathcal{B})$ and whose morphism spaces are defined by the subspaces

$$\mathsf{Hom}_{\mathcal{N}_\infty(\mathcal{A},\mathcal{B})}(M, M') = \mathsf{Hom}_{\mathcal{C}_\infty(B^+\mathcal{A},B^+\mathcal{B})}(BM, BM'), \quad M, M' \in \mathcal{C}_\infty(\mathcal{A},\mathcal{B}).$$

If $\mathcal{A}$ and $\mathcal{B}$ are strictly unital, we define the categories $\big(\mathcal{N}_\infty(\mathcal{A},\mathcal{B})\big)_u$ and $\mathcal{C}_\infty(\mathcal{A},\mathcal{B})$ analogously to the categories $\big(\mathcal{N}_\infty\mathcal{A}\big)_u$ and $\mathcal{C}_\infty\mathcal{A}$. The category $\mathsf{Mod}_\infty(\mathcal{A},\mathcal{B})$ is isomorphic to $Z^0\mathcal{C}_\infty(\mathcal{A},\mathcal{B})$.

Proposition 5.2.0.4. The inclusion

$$\mathcal{C}_\infty(\mathcal{A},\mathcal{B}) \to \mathcal{N}_\infty(\mathcal{A},\mathcal{B})$$

induces a quasi-isomorphism on the morphism spaces. □

5.3 Key lemma

The lemma below will be useful for the construction of the Yoneda A_∞-functor (see Definition 7.1.0.1).

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets, M a graded object of $\mathsf{C}(\mathbb{A},\mathbb{B})$ and $\mathcal{A}$ and $\mathcal{B}$ two A_∞-categories on $\mathbb{A}$ and $\mathbb{B}$. Consider a family of graded morphisms of $\mathbb{A}$-$\mathbb{B}$-bimodules

$$m_{i,j} : \mathcal{A}^{\odot_{\mathbb{A}} i} \odot_{\mathbb{A}} M \odot_{\mathbb{B}} \mathcal{B}^{\odot_{\mathbb{B}} j} \to M, \quad i, j \geq 0,$$

of degree $1-i-j$. We endow the co-augmented tensor coalgebras $T^cS\mathcal{B}$ and $T^cS\mathcal{A}$ with the differentials $b^{\mathcal{A}}$ and $b^{\mathcal{B}}$ of the co-augmented bar constructions. The morphisms

$$b_{0,j} : SM \odot_{\mathbb{B}} (S\mathcal{B})^{\odot_{\mathbb{B}} j} \to SM, \quad j \geq 0,$$

given by the bijections $m_{0,j} \leftrightarrow b_{0,j}$ of the section 2.5.1, can be lifted (see Lemma 2.1.2.1) into a unique coderivation of graded comodules in $\mathsf{C}(\mathbb{A},\mathbb{B})$

$$D : SM \odot_{\mathbb{B}} T^c(S\mathcal{B}) \to SM \odot_{\mathbb{B}} T^c(S\mathcal{B}).$$

Let $\mathsf{End} = \mathsf{End}\Big(SM \odot_{\mathbb{B}} T^c(S\mathcal{B})\Big)$ be the algebra of graded endomorphisms of $B^+\mathcal{B}$-comodules in the category $C(\mathbb{A},\mathbb{B})$. Note that this object of the category $\mathsf{C}(\mathbb{A},\mathbb{A})$ is also defined by

$$\mathsf{End}(A, A') = \overset{\infty}{\mathsf{Hom}}_{\mathcal{B}}\big(M, M(-, A')\big)(A), \quad A, A' \in \mathbb{A},$$

where $\overset{\infty}{\mathsf{Hom}}$ is the functor defined in (Section 4.1.1). We endow End with the three morphisms

$$\begin{aligned} m_0 : e_{\mathbb{A}} &\to \mathsf{End}, & 1 &\mapsto -D^2 \\ m_1 : \mathsf{End} &\to \mathsf{End}, & f &\mapsto D \circ f - (-1)^r f \circ D \\ m_2 : \mathsf{End} \odot_{\mathbb{A}} \mathsf{End} &\to \mathsf{End}, & f \odot g &\mapsto f \circ g, \end{aligned}$$

where f is a morphism of degree r. They satisfy the equations

$$\begin{aligned} m_1 \cdot m_0 &= 0, \\ m_2(m_0 \odot \mathbf{1} + \mathbf{1} \odot m_0) + m_1^2 &= 0, \\ m_2(m_1 \odot \mathbf{1} + \mathbf{1} \odot m_1) - m_1 m_2 &= 0, \end{aligned}$$

and

$$m_2(\mathbf{1} \odot m_2 - m_2 \odot \mathbf{1}) = 0.$$

A differential graded algebra (A, d, μ) clearly satisfies these equations for $m_0 = 0$, $m_1 = d$ and $m_2 = \mu$. Conversely, if M is a graded object endowed with morphisms m_0, m_1 and m_2 satisfying these equations, (M, m_1, m_2) is a differential graded algebra if m_0 is zero.

Let the graded morphisms of $\mathbb{A}$-$\mathbb{A}$-bimodules

$$f_i : \mathcal{A}^{\odot i} \to \mathsf{End}, \quad i \geq 1,$$

of degree $2 - i$, defined by the equation

$$F_i(\phi) = s(\Phi) \in S\mathsf{End},$$

where F_i is given by the bijection $f_i \leftrightarrow F_i$, where ϕ is an element of $(S\mathcal{A})^{\odot i}$ of degree $|\phi|$ and where the morphism Φ is the unique morphism (see Lemma 2.1.2.1) such that the composition $p_1 \circ \Phi$ has as components the morphisms

$$SM \odot (S\mathcal{B})^{\odot j} \xrightarrow{(-1)^{|\phi|}\phi\odot\mathbf{1}} (S\mathcal{A})^{\odot i} \odot SM \odot (S\mathcal{B})^{\odot j} \xrightarrow{b_{i,j}} SM, \quad j \geq 0.$$

Lemma 5.3.0.1. The following statements are equivalent.

a. The triple (End, m_1, m_2) is a differential graded algebra and the morphisms f_i, $i \geq 1$, define an A_∞-morphism
$$f : \mathcal{A} \to \mathsf{End},$$
where End is equipped with the A_∞-structure of Remark 1.2.1.5.

b. The morphisms $m_{i,j}$, $i, j \geq 0$, define the structure of an $\mathcal{A}$-$\mathcal{B}$-bipolydule on M.

Proof. Suppose the statement a is true. We will show that it is equivalent to the equations

$$\sum_{k+\bullet+m=n} b_\bullet(\mathbf{1}^{\odot k} \odot b_\bullet \odot \mathbf{1}^{\odot m}) = 0, \quad n \geq 0,$$

where $b_\bullet$ symbols should be interpreted appropriately. These equations are equivalent to the equations $(*''_n)$, $n + 1 + n' \geq 1$, of Definition 2.5.1.3.

As (End, m_1, m_2) is a differential graded algebra, the morphism m_0 is zero. This means that D is a comodule differential. The equation $D^2 = 0$ is equivalent to the equations

$$\sum_{1+j+k=n} b_{0,l}(b_{0,j} \odot \mathbf{1}^{\odot k}) + \sum_{k+j+m=n} b_{0,l}(\mathbf{1}^{\odot k} \odot b_j^{\mathcal{B}} \odot \mathbf{1}^{\odot m}) = 0, \quad n \geq 0.$$

By virtue of Section 1.2.2, the fact that f is an A_∞-morphism results in the fact that the sequence of morphisms F_i, $i \geq 1$, defines a morphism of differential graded coalgebras

$$F : B^+\mathcal{A} \to B^+\mathsf{End}.$$

This is equivalent to the equations $(**_n)$, $n \geq 1$:

$$\sum_{i+j+k=n} F_l(\mathbf{1}^{\odot i} \odot b_j^{\mathcal{A}} \odot \mathbf{1}^{\odot k}) - b_1^{\mathsf{End}}(F_n) - \sum_{i+j=n} b_2^{\mathsf{End}}(F_i \odot F_j) = 0.$$

We recall that the definition of the bijections $m_i^{\mathsf{End}} \leftrightarrow b_i^{\mathsf{End}}$, $i \geq 2$, implies that

$$b_1^{\mathsf{End}} \circ s = -s \circ m_1^{\mathsf{End}} \quad \text{and} \quad b_2^{\mathsf{End}} \circ s^{\odot 2} = s \circ m_2^{\mathsf{End}}.$$

Let $m \odot y$ be an element of $SM \odot (S\mathcal{B})^{\odot n}$. Compute the image of $m \odot y$ by $b_2^{\mathsf{End}}(F_i \odot F_j)(\phi)$ where $\phi = \phi_j \odot \phi_i$:

$$\begin{aligned}
b_2^{\mathsf{End}}(F_i \odot F_j)(\phi)(m \odot y) &= b_2^{\mathsf{End}}(s\Phi_i \odot s\Phi_j)(m \odot y) \\
&= (-1)^{|\Phi_i|} b_2^{\mathsf{End}}(s \odot s)(\Phi_i \odot \Phi_j)(m \odot y) \\
&= (-1)^{|\Phi_i|} s m_2^{\mathsf{End}}(\Phi_i \odot \Phi_j)(m \odot y) \\
&= \sum_{k+l=n} (-1)^{|\Phi_i|+|\phi_i|+|\phi_j|} s b_{i,l}(\phi_i \odot b_{j,k}(\phi_j \odot \mathbf{1}_{SM} \odot \mathbf{1}^{\otimes k}) \odot \mathbf{1}^{\odot l})(m \odot y) \\
&= \sum_{k+l=n} (-1)^{|\phi_j|+1} s b_{i,l}(\phi_i \odot b_{j,k}(\phi_j \odot \mathbf{1}_{SM} \odot \mathbf{1}^{\odot k}) \odot \mathbf{1}^{\odot l})(m \odot y) \\
&= \sum_{k+l=n} (-1)^{|\phi|+1} s b_{i,l}(\mathbf{1}_{S\mathcal{A}}^{\odot i} \odot b_{j,j}(\mathbf{1}_{S\mathcal{A}}^{\odot j} \odot \mathbf{1}_{SM} \odot \mathbf{1}^{\odot k}) \odot \mathbf{1}^{\otimes l})(\phi \odot m \odot y),
\end{aligned}$$

then $b_1^{\mathsf{End}}(F_n)(\phi)(m \odot y)$:[1]

$$\begin{aligned}
b_1^{\mathsf{End}}(F_n)(\phi)(y) &= b_1^{\mathsf{End}}(s\Phi)(m \odot y) \\
&= -s m_1^{\mathsf{End}}(\Phi)(m \odot y) \\
&= -s(b \cdot \Phi - (-1)^{|\Phi|}\Phi \cdot b)(m \odot y) \\
&= -s\Bigg[\sum_{k+l=n} (-1)^{|\phi|} b_{0,l}(b_{i+j,k}(\phi \odot \mathbf{1}_{SM} \odot \mathbf{1}^{\odot k}) \odot \mathbf{1}^{\odot l}) + \\
&\qquad + \sum_{u+v+l=n} -(-1)^{|\phi|} b_{i+j,u+1+l}(\phi \odot \mathbf{1}_{SM} \odot \mathbf{1}^{\odot u} \odot b_v^{\mathcal{B}} \odot \mathbf{1}^{\odot l}) + \\
&\qquad - (-1)^{|\phi|} b_{i+j,u+1+l}(\phi \odot b_{0,n}(\mathbf{1}_{SM} \odot \mathbf{1}^{\odot n}) \Bigg](m \odot y) \\
&= (-1)^{|\phi|+1} s\Bigg[\sum_{k+l=n} b_{0,l}(b_{i+j,k}(\mathbf{1}_{S\mathcal{A}}^{\odot i+j} \odot \mathbf{1}_{SM} \odot \mathbf{1}^{\odot k}) \odot \mathbf{1}^{\odot l}) + \\
&\qquad + \sum_{u+v+l=n} b_{i+j,u+1+l}(\mathbf{1}_{S\mathcal{A}}^{\odot i+j} \odot \mathbf{1}_{SM} \odot \mathbf{1}^{\odot u} \odot b_v^{\mathcal{B}} \odot \mathbf{1}^{\odot l}) + \\
&\qquad + b_{i+j,u+1+l}(\mathbf{1}_{S\mathcal{A}}^{\odot i+j} \odot b_{0,n}(\mathbf{1}_{SM} \odot \mathbf{1}^{\odot n}) \Bigg](\phi \odot m \odot y)
\end{aligned}$$

and finally $F_l(\mathbf{1}^{\odot i} \odot b_j^{\mathcal{A}} \odot \mathbf{1}^{\odot k})(\phi)(m \odot y)$:

$$\begin{aligned}
&F_l(\mathbf{1}^{\odot i} \odot b_j^{\mathcal{A}} \odot \mathbf{1}^{\odot k})(\phi)(m \odot y) = \\
&\qquad \sum_{u+v+t=i+j} (-1)^{|\phi|+1} b_{u+1+t,n}(\mathbf{1}_{S\mathcal{A}}^{\odot u} \odot b_v^{\mathcal{A}} \odot \mathbf{1}_{S\mathcal{A}}^{\odot t} \odot \mathbf{1}_{SM} \odot \mathbf{1}^{n})(\phi \odot m \odot y).
\end{aligned}$$

[1] Some parens are missing in the following. Presently unable to sanity check the expressions.

The equations $(**_n)$, $n \geq 1$, and the fact that the coderivation D is a differential are therefore equivalent to the equations

$$\sum b_\bullet(\mathbf{1}^{\odot u} \odot b_\bullet \odot \mathbf{1}^{\odot v}) = 0$$

where the $b_\bullet$ and the $\mathbf{1}$ must be interpreted appropriately. □

Let M be a $\mathcal{A}$-$\mathcal{B}$-bipolydule. Let A be an object of $\mathcal{A}$. We endow the $\mathbb{A}$-module $M(-,A)$ with the structure of a $\mathcal{B}$-polydule given by the morphisms m_j, $j \geq 1$, of $\mathbb{B}$-modules

$$m_{0,j-1}(-,A) : \Big(M \odot \mathcal{B}^{\otimes j-1}\Big)(-,A) \to M(-,A), \quad j \geq 1.$$

Corollary 5.3.0.2. The map

$$\dot{\theta}_M : \mathbb{A} \to \mathsf{Obj}\,\mathcal{N}_\infty\mathcal{B}, \quad A \mapsto M(-,A)$$

can be canonically extended to an A_∞-functor

$$\theta_M : \mathcal{A} \to \mathcal{N}_\infty\mathcal{B}.$$

Proof. The $\mathbb{A}$-$\mathbb{A}$-bimodule

$$\mathsf{Hom}_{\mathcal{N}_\infty\mathcal{B}}(\dot{\theta_M}-, \dot{\theta_M}-)$$

is by definition the endomorphism algebra

$$\mathsf{End}_{\mathcal{N}_\infty B^+\mathcal{B}}\big((?,-)\otimes T^c S\mathcal{B}\big),$$

that is, the algebra End of the key lemma. The functor canonically associated with M is given by the morphisms

$$f_i : \mathcal{A}^{\odot i} \to \mathsf{Hom}_{\mathcal{N}_\infty\mathcal{B}}(\dot{\theta_M}-, \dot{\theta_M}-), \quad i \geq 1,$$

of Lemma 5.3.0.1. They define an A_∞-functor because

$$f : \mathcal{A} \to \mathsf{End}$$

is an A_∞-morphism. □

Corollary 5.3.0.3. The map $M \mapsto \theta_M$ from the class of $\mathcal{A}$-$\mathcal{B}$-bipolydules to the class of A_∞-functors $\mathcal{A} \to \mathcal{N}_\infty\mathcal{B}$ is a bijection. Its inverse map associates to an A_∞-functor

$$(\dot{g}, g) : \mathcal{A} \to \mathcal{N}_\infty\mathcal{B}$$

the $\mathbb{A}$-$\mathbb{B}$-bimodule

$$M(A,B) = (\dot{g}(A))(B)$$

endowed with the multiplications $m_{i,j}$, $i,j \geq 0$, given by

$$m_{i,j-1} = (g_i)_j.$$

□

The strictly unital case

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are strictly unital A_∞-categories.

Remark 5.3.0.4. Let M be an $\mathcal{A}$-$\mathcal{B}$-bipolydule. The A_∞-morphism

$$f : \mathcal{A} \to \mathsf{End}$$

of the key lemma (5.3.0.1) is strictly unital if and only if the compositions

$$m^M_{i,j}(\mathbf{1}^{\odot\alpha} \odot \eta \odot \mathbf{1}^{\odot\beta} \otimes \mathbf{1}_M \otimes \mathbf{1}^{\otimes j}), \quad i, j \geq 0,$$

are zero for $(i,j) \neq (1,0)$ and are the identity for $(i,j) = (1,0)$.

Remark 5.3.0.5. If M is a strictly unital $\mathcal{A}$-$\mathcal{B}$-bipolydule, the $\mathcal{B}$-polydule $M(-, A)$, $A \in \mathbb{A}$, is strictly unital and the A_∞-function

$$\mathcal{A} \to \mathcal{N}_\infty\mathcal{B}$$

of Corollary (5.3.0.2) is factorized by a functor

$$\mathcal{A} \to \mathcal{C}_\infty\mathcal{B}.$$

Remark 5.3.0.6. The bijection $M \mapsto \theta_M$ of Corollary (5.3.0.3) is restricted to a bijection from the class of strictly unital $\mathcal{A}$-$\mathcal{B}$-bipolydules to the class of strictly unital A_∞-functors $\mathcal{A} \to \mathcal{C}_\infty\mathcal{B}$.

Chapter 6

Torsion of A_∞-structures

In Chapters 7 and 8, we will construct A_∞-categories whose compositions are built using a process of torsion described in this chapter

In the theory of deformations of differential graded Lie algebras (or differential graded associative algebras), the technique of torsion is well known (for an overview, see, for example, [Hue99]). The A_∞ (and L_∞) version was introduced in [FOOO01, Chap. 4] (see also [Fuk01a]). Our proof that the twisted compositions define an A_∞ structure is different. The torsion of an A_∞-algebra A by a solution to the generalized Maurer-Cartan equation not only modifies the differential m_1 but also all the higher multiplications.

This chapter is divided into two sections. We first deal with the simple case where the torsion is tensorially nilpotent, and then the case where the A_∞ structures are topological. We show that if $f : \mathcal{A} \to \mathcal{B}$ is an A_∞-functor that induces quasi-isomorphisms in the spaces of morphisms, its torsion also induces quasi-isomorphisms in the spaces of morphisms (6.1.3.4).

6.1 The tensorially nilpotent case

6.1.1 Twisting elements

Let $\mathbb{A}$ be a set and $\mathcal{A}$ be an A_∞-category over $\mathbb{A}$. Equip the neutral element $e = e_{\mathbb{A}}$ for the tensor product $\odot_{\mathbb{A}}$ with the coalgebra structure provided by the unitality constraint of the base monoidal category $\mathsf{C}(\mathbb{A}, \mathbb{A})$ (see 5.1.1 and 1.1.1).

$$e \xrightarrow{\sim} e \odot e.$$

Consider e as a differential graded coalgebra concentrated in degree 0.

Definition 6.1.1.1. A *twisted (tensorially nilpotent) element* is a graded morphism $x : e \to \mathcal{A}$ of degree $+1$ such that

(1) the composite $e \circ x$ can be lifted to a morphism of coalgebras

$$X : e \to B\mathcal{A},$$

(2) and the morphism X is compatible with differentials.

Remark 6.1.1.2. Let p_1 denote the projection $B\mathcal{A} \to S\mathcal{A}$. The composition with the projection yields a bijection

$$\mathsf{Hom}_{\mathsf{Cog}}(e, B\mathcal{A}) \xrightarrow{\sim} \mathsf{Hom}_{nil}(e, S\mathcal{A}),$$

where $\mathsf{Hom}_{nil}(e, S\mathcal{A})$ is the set of graded morphisms $\phi : e \to S\mathcal{A}$ of degree 0 such that, for all $A \in \mathbb{A}$, there exists an N such that $\phi^{\otimes n}\Delta^{n-1}(\mathbf{I}_A) = 0$ for $n \geq N$. We conclude from this that a graded morphism $x : e \to \mathcal{A}$ of degree $+1$ is a twisting element if and only if

(1) it is *tensorially nilpotent*: for any object $a \in \mathbb{A}$, the element $x(\mathbf{I}_A) \in \mathcal{A}(A, A)$ of degree 1 is such that $x(\mathbf{I}_A)^{\odot n}$ is zero for some $n > 0$,

(2) it satisfies the Maurer-Cartan equation

$$\sum_{i=1}^{\infty} m_i\big(x(\mathbf{I}_A) \odot \ldots \odot x(\mathbf{I}_A)\big) = 0, \quad A \in \mathbb{A}.$$

(The sum is finite due to the tensorial nilpotence property).

6.1.2 Torsion of A$_\infty$-categories

Let $\mathbb{A}$ be a set and $\mathcal{A}$ be an A$_\infty$-category over $\mathbb{A}$. Let x be a tensorially nilpotent element in $\mathcal{A}$. Let

$$g : T^c S\mathcal{A} = e \oplus \overline{T^c} S\mathcal{A} \to S\mathcal{A}$$

the morphism of components $[sx, p_1]$, where p_1 is the projection $\overline{T^c}S\mathcal{A} \to S\mathcal{A}$.

Consider the morphism of $\mathbb{A}$-$\mathbb{A}$-bimodules

$$\phi_x : T^c S\mathcal{A} \to T^c S\mathcal{A} = \bigoplus_{i \geq 0} (S\mathcal{A})^{\odot i}$$

whose composition with the projection to $(S\mathcal{A})^{\odot i}$ is the morphism

$$(g^{\odot i}) \circ \Delta^{(i)} \quad \text{if} \quad i \geq 1, \quad \mathbf{1}_e \quad \text{otherwise.}$$

It is clearly a co-unital co-algebra morphism, and it is well-defined because its restriction to the subobject $(S\mathcal{A})^{\odot i} \in \mathsf{C}(\mathbb{A}, \mathbb{A})$ is equal to the sum (well-defined by the tensorial nilpotence property).

$$\sum_{l \geq 0} \sum \big((sx)^{\odot l_0} \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_1} \odot \ldots \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_{i-1}} \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_i}\big),$$

where $l_0 + \ldots + l_i = l$. Note that the composition

$$\phi_x \circ \varepsilon : e \to T^c S\mathcal{A} = e \oplus \overline{T^c} S\mathcal{A},$$

where ε is the co-augmentation of $T^c S\mathcal{A}$, has components equal to the morphism $\mathbf{1}_e$ and the lift X of $s \circ x$. The matrix of

$$\phi_x : \bigoplus_{j \geq 0} (S\mathcal{A})^{\odot j} \to \bigoplus_{i \geq 0} (S\mathcal{A})^{\odot i}$$

is lower triangular, and its diagonal is the same as that of the identity. The morphism ϕ_x is therefore a co-unital automorphism (not co-augmented) of the co-augmented graded coalgebra $T^c S\mathcal{A}$. The differential of the bar construction $B\mathcal{A}$ gives us a differential

$$b : T^c S\mathcal{A} \to T^c S\mathcal{A}$$

that vanishes on the co-augmentation. Consider the composition

$$D_x = \phi_x^{-1} \circ b \circ \phi_x : T^c S\mathcal{A} \to T^c S\mathcal{A}.$$

Suppose x satisfies the Maurer-Cartan equation. The lift $X : e \to \overline{T^c} S\mathcal{A}$ of $s \circ x$ is differential graded. The composition

$$b \circ \phi_x \circ \varepsilon = b \circ \begin{bmatrix} \mathbf{1}_e \\ X \end{bmatrix} : e \to T^c S\mathcal{A} = e \oplus \overline{T^c} S\mathcal{A}$$

is therefore zero, and we have $D_x \circ \varepsilon = 0$. Let b_x be the morphism given by the rightmost vertical arrow in the diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & e \arrow[r, "\varepsilon"] \arrow[d, "0"'] & T^c S\mathcal{A} \arrow[r] \arrow[d, "D_x"'] & \overline{T^c} S\mathcal{A} \arrow[r] \arrow[d, "b_x"'] & 0 \\ 0 \arrow[r] & e \arrow[r, "\varepsilon"] & T^c S\mathcal{A} \arrow[r] & \overline{T^c} S\mathcal{A} \arrow[r] & 0. \end{tikzcd}$$

Since D_x is a $(\mathbf{1}, \mathbf{1})$-coderivation of $T^c S\mathcal{A}$, the morphism b_x is an $(\mathbf{1}, \mathbf{1})$-coderivation of $\overline{T^c} S\mathcal{A}$. As $D_x^2 = 0$, the coderivation b_x is a differential of the coalgebra $\overline{T^c} S\mathcal{A}$. It is determined (1.1.2.2) by the components

$$(b_x)_i : (S\mathcal{A})^{\odot i} \to S\mathcal{A}$$

of its composition with the projection to $S\mathcal{A}$.

Lemma 6.1.2.1. Let $i \geq 1$. The morphism $(b_x)_i$ is the sum

$$\sum_l \sum b_{l+m}((sx)^{\odot l_0} \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_1} \odot \ldots \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_{i-1}} \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_i}),$$

where $l_0 + \ldots + l_i = l$.

Proof. We remark that D_x restricted to the sub-object $\overline{T^c} S\mathcal{A}$ of $T^c S\mathcal{A}$ is equal to b_x. We need to calculate

$$(p_1 \circ D_x)|_{(S\mathcal{A})^{\odot i}} = (p_1 \circ \phi_x^{-1} \circ b \circ \phi_x)|_{(S\mathcal{A})^{\odot i}} \quad i \geq 1.$$

Since the matrix of coefficients

$$\phi_x : \bigoplus_{j \geq 0} (S\mathcal{A})^{\odot j} \to \bigoplus_{i \geq 0} (S\mathcal{A})^{\odot i}$$

is lower triangular and its diagonal is that of the identity, the matrix of ϕ_x^{-1} has the same form. Thus, the morphisms $p_1 \circ \phi_x^{-1} \circ b \circ \phi_x$ and $p_1 \circ b \circ \phi_x$ restricted to $(S\mathcal{A})^{\odot i}$ are equal. This proves the lemma. $\square$

Definition 6.1.2.2 (K. Fukaya [FOOO01] (see also [Fuk01a])). A *twisted* A_∞*-category* $\mathcal{A}_x$ over $\mathbb{A}$ is a $\mathbb{A}$-$\mathbb{A}$-bimodule $\mathcal{A}_x = \mathcal{A}$ whose bar construction $B\mathcal{A}_x$ is the reduced differential graded tensor coalgebra

$$(\overline{T^c} S\mathcal{A}, b_x).$$

The compositions

$$m_i^x : \mathcal{A}_x^{\odot i} \to \mathcal{A}_x, \quad i \geq 1,$$

are defined by the sum

$$\sum_l \sum (-1)^s m^{\mathcal{A}}_{l+i}(x^{\odot l_0} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_1} \ tso \ldots \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_{i-1}} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_i}),$$

where the exponent is $s = \sum_{1 \le t \le i} t \times l_t$. (This infinite sum indeed defines a morphism thanks to the tensor-nilpotence property of x.)

6.1.3 Twisting of A$_\infty$-functors

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets, and let $\mathcal{A}$ and $\mathcal{B}$ be two A$_\infty$-categories over $\mathbb{A}$ and $\mathbb{B}$, respectively. Let

$$(\dot{f}, f) : \mathcal{A} \to \mathcal{B}$$

be an A$_\infty$-functor and x and x' be torsion elements in $\mathcal{A}$ and $\mathcal{B}$ satisfying a compatibility relation with f that will be specified later. This relation roughly says that the image of x under f is x'. The goal of this section is to construct a twisted A$_\infty$-functor

$$\mathcal{A}_x \to \mathcal{B}_{x'}.$$

Equip the $\mathbb{A}$-$\mathbb{A}$-bimodule ${}_{\dot{f}}\mathcal{B}_{\dot{f}}$ with the structure of an A$_\infty$-category over $\mathbb{A}$ from Lemma 5.1.2.4. Let $\mathcal{B}'$ denote this A$_\infty$-category over $\mathbb{A}$. The twisting element

$$x' : e_{\mathbb{B}} \to \mathcal{B}$$

induces a twisting element in $\mathcal{B}'$

$$e_{\mathbb{A}} \to \mathcal{B}', \quad \mathbf{I}_A \mapsto x'(\dot{f}A).$$

We also denote this as x'. Let

$$F : B^+\mathcal{A} \to B^+\mathcal{B}'$$

be the co-augmented bar construction of the A$_\infty$-morphism $f : \mathcal{A} \to \mathcal{B}'$. We will construct the twisted A$_\infty$-functor in such a way that the morphism

$$G = \phi_{x'}^{-1} \circ F \circ \phi_x : T^c S\mathcal{A} \longrightarrow T^c S\mathcal{B}'$$

is its coaugmented bar construction. Note that for any twisting elements x and x', the morphism G is indeed a differential graded morphism

$$G : B^+\mathcal{A}_x \to B^+\mathcal{B}'_{x'}.$$

However, there is no guarantee that it is co-augmented, as ϕ_x and $\phi_{x'}$ are not co-augmented. Demanding that it be co-augmented leads to compatibility relations between x and x': Suppose G is augmented. We have the equation

$$\phi_{x'}^{-1} \circ F \circ \phi_x \circ \varepsilon = \varepsilon,$$

or in other words, we have

$$F \circ \phi_x \circ \varepsilon = \phi_{x'} \circ \varepsilon.$$

Since the compositions $\phi_x \circ \varepsilon$ and $\phi_{x'} \circ \varepsilon$ are equal to the maps

$$\mathbf{1}_e + X : e \to e \oplus \overline{T^c} S\mathcal{A} \quad \text{and} \quad \mathbf{1}_e + X' : e \to e \oplus \overline{T^c} S\mathcal{B}'$$

where X and X' are the lifts of $x : e \to \mathcal{A}$ and $x' : e \to \mathcal{B}'$, the compatibility between x and x' asserts that the sum (well-defined due to the tensorial nilpotentence property of x)

$$\sum_{i \geq 1} f_i(x^{\odot i}) : e \to \mathcal{B}'$$

is equal to the twisting object x'.

Since the map G is co-augmented, it is the co-augmentation of a morphism of reduced differential graded coalgebras.

$$F_x : B\mathcal{A}_x \to B\mathcal{B}'_{x'}.$$

Lemma 6.1.3.1. Let $i \geq 1$. The morphism $(F_x)_i : (S\mathcal{A})^{\odot i} \to S\mathcal{B}'$ is the sum

$$\sum_l \sum F_{l+m}((sx)^{\odot l_0} \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_1} \odot \ldots \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_{i-1}} \odot \mathbf{1}_{S\mathcal{A}} \odot (sx)^{\odot l_i}),$$

where $l_0 + \ldots + l_i = l$.

Proof. Similar to the one in Lemma 6.1.2.1. □

Note that the A_∞-category $\mathcal{B}'_{x'}$ is equal to ${}_{\dot{f}}(\mathcal{B}_{x'})_{\dot{f}}$.

Definition 6.1.3.2. The *twisted* A_∞*-functor*

$$(\dot{f}, f^x) : \mathcal{A}_x \to \mathcal{B}_{x'}$$

is the functor whose bar construction is F_x.

It is defined by the morphisms

$$f_i^x : \mathcal{A}_x^{\odot i} \to {}_{\dot{f}}(\mathcal{B}_{x'})_{\dot{f}}, \quad i \geq 1,$$

defined by the sums

$$\sum_l \sum (-1)^s f_{l+i}^{\mathcal{A}}(x^{\odot l_0} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_1} \odot \ldots \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_{i-1}} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_i}),$$

where the exponent of the sign is $s = \sum_{1 \leq t \leq i} t \times l_t$.

Torsion and weak equivalences

Lemma 6.1.3.3. Let $\mathcal{A}$ be an A_∞-category and x a tensorially nilpotent twisting element. Let $\mathcal{A}$ be an A_∞-category weakly equivalent to zero, i.e. the morphism in $\mathsf{C}(\mathbb{A}, \mathbb{A})$

$$\mathcal{A} \to 0$$

is an A_∞-quasi-isomorphism. The twisted category $\mathcal{A}_x$ is weakly equivalent to zero.

Proof. The ambient category for the reasoning below is $\mathsf{C}(\mathbb{A},\mathbb{A})$. We recall (5.1.2.7) that an A$_\infty$-morphism f between two A$_\infty$-algebras in $\mathsf{C}(\mathbb{A},\mathbb{A})$ is an A$_\infty$-functor whose underlying map $\dot{f}$ is the identity in $\mathbb{A}$. Let K be the contractible complex $(\mathcal{A}, m_1)$. Consider it as an A$_\infty$-algebra (1.2.1.4). Lemma (1.3.3.2) shows that there exists an A$_\infty$-(iso)morphism

$$f : \mathcal{A} \to K$$

such that $f_1 = \mathbf{1}_K$. Consider the A$_\infty$-algebra K with the twisting element

$$x' = \sum_{i\geq 1} f_i(x^{\odot i}).$$

The commutative diagram

$$\begin{tikzcd} B^+\mathcal{A} \arrow[r, "F"] \arrow[d, "\phi_x"'] & B^+K \arrow[d, "\phi_{x'}"] \\ B^+\mathcal{A}_x \arrow[r, "G"'] & B^+K_{x'} \end{tikzcd}$$

shows that G is an isomorphism. In particular, $\mathcal{A}_x$ is A$_\infty$-quasi-isomorphic to $K_{x'}$. It suffices to show that $K_{x'}$ is weakly equivalent to zero. By construction, the multiplications m_i^K for $i \geq 2$ are zero. This implies that

$$m_1^{K_{x'}} = m_1^K \quad \text{and} \quad m_i^{K_{x'}} = 0 \quad i \geq 2.$$

Therefore, the twisted A$_\infty$-category $K_{x'}$ is equal to K and is weakly equivalent to zero. $\square$

Proposition 6.1.3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be A$_\infty$-categories over $\mathbb{A}$ and $\mathbb{B}$. Let

$$(\dot{f}, f) : \mathcal{A} \to \mathcal{B}$$

be an A$_\infty$-functor that induces a quasi-isomorphism on morphism spaces, i.e. the morphisms

$$f_1 : \mathcal{A}(A, A') \to \mathcal{B}(\dot{f}A, \dot{f}A'), \quad A, A' \in \mathbb{A},$$

are quasi-isomorphisms. Let x and x' be nilpotent twisting elements in $\mathcal{A}$ and $\mathcal{B}$ compatible with f. The twisted A$_\infty$-functor

$$(\dot{f}, f^x) : \mathcal{A}_x \to \mathcal{B}_{x'}$$

induces a quasi-isomorphism on the morphism spaces.

Proof. Note that $\mathcal{B}'$ is the A$_\infty$-category ${}_{\dot{f}}\mathcal{B}_{\dot{f}}$ over $\mathbb{A}$ (see 5.1.2.4). The A$_\infty$-functor f induces a quasi-isomorphism in the morphism spaces if and only if the A$_\infty$-morphism in the category of A$_\infty$-algebras in $\mathsf{C}(\mathbb{A},\mathbb{A})$

$$f' : \mathcal{A} \to \mathcal{B}'$$

induced by f is a weak equivalence. Therefore, suppose that f is an A$_\infty$-quasi-isomorphism in $\mathsf{C}(\mathbb{A},\mathbb{A})$. The proof of the factorization axiom (CM5) *a.* of the category Alg_∞ (1.3.3.1) gives us a factorization of f into

$$\mathcal{A} \rightarrowtail^{i} \mathcal{A}\prod C \twoheadrightarrow \mathcal{B},$$

where $\mathcal{A}\prod C$ is the product in Alg_∞ of $\mathcal{A}$ with the cone C of the identity of the complex $(\mathcal{B}, m_1)$ (considered as an A$_\infty$-algebra), and i has components $\mathbf{1}_\mathcal{A}$ and 0. It is sufficient to show the result in the case where f is equal to i and in the case where it is a trivial fibration. Let's start with the trivial cofibration i. Equipping $\mathcal{A}\prod C$ with the twisting element

$$x'' = \sum_{j\geq 1} i_j(x^{\odot j}).$$

We have the equalities

$$(\mathcal{A}\prod C)_{x''} = \mathcal{A}_x \prod C \quad \text{and} \quad i^x = \begin{bmatrix} \mathbf{1}_{\mathcal{A}_x} \\ 0 \end{bmatrix} : \mathcal{A}_x \to \mathcal{A}_x \prod C.$$

As a result, i^x is a weak equivalence. Now, suppose that f is a trivial fibration. A splitting of f_1 in the category of complexes gives us an isomorphism of complexes

$$j : \mathcal{A} \to \mathcal{B} \oplus K,$$

where K is a contractible complex. Let $\mathcal{B}\prod K$ be the product in Alg_∞ of the A_∞-algebra $\mathcal{B}$ and the complex K considered as an A_∞-algebra. The canonical projection $p : \mathcal{B}\prod K \to \mathcal{B}$ is a trivial fibration. Remark (1.3.3.4) applied to the lifting axiom (CM4) a gives us an A_∞-isomorphism

$$\tilde{f} : \mathcal{A} \to \mathcal{B}\prod K$$

such that $\tilde{f}_1 = j$ and $p \circ \tilde{f} = f$. Equipping $\mathcal{B}\prod K$ with the twisting element

$$x'' = \sum_{j\geq 1} \tilde{f}_j(x^{\odot j}).$$

We have the equality

$$(\mathcal{B}\prod K)_{x''} = \mathcal{B}_{x'} \prod K$$

and the twisted A_∞-morphism $p^{x''}$ corresponds to the canonical projection

$$\mathcal{B}_{x'} \prod K \to \mathcal{B}_{x'}.$$

Since K is contractible, $p^{x''}$ is a weak equivalence. The equality $f^x = p^{x''} \circ \tilde{f}^x$ shows that f^x is a weak equivalence. $\square$

6.1.4 Torsion of $\mathcal{A}$-$\mathcal{B}$-bipolydules

The details are omitted as they are similar to the last two sections.

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets, $\mathcal{A}$ and $\mathcal{B}$ be two A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$ and M be a $\mathcal{A}$-$\mathcal{B}$-bipolydule. Let x and x' be twisting elements in $\mathcal{A}$ and $\mathcal{B}$.

Definition 6.1.4.1. A $\mathcal{A}_x$-$\mathcal{B}_{x'}$*-bipolydule* ${}_xM_{x'}$ has multiplication morphisms

$$m^{x,x'}_{i,j} : \mathcal{A}_x^{\odot i} \odot {}_xM_{x'} \odot \mathcal{B}_{x'} \to {}_xM_{x'}, \quad i,j \geq 0,$$

defined by the sum

$$\sum_{l,k\geq 0} \sum (-1)^s m_{i+l,j+k}(x^{\odot l_0} \odot \mathbf{1}_{\mathcal{A}} \ldots \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_i} \odot \mathbf{1}_M \odot x'^{\odot k_o} \odot \mathbf{1}_{\mathcal{B}} \ldots \mathbf{1}_{\mathcal{B}} \odot x'^{\odot k_j}),$$

whose exponent has sign

$$s = \Big(\sum_{1 \leq t \leq i} t \times l_t \Big) + \Big(\sum_{1 \leq t \leq j} (j+t) \times l_t \Big)$$

(The infinite sums make well-defined morphisms thanks to the tensorial nilpotence property of x and x').

Remark 6.1.4.2. The differential $b_{x,x'}$ of the bar construction of the $\mathcal{A}_x$-$\mathcal{B}_{x'}$-bipolydule ${}_xM_{x'}$ is given by the composite

$$\big(\phi_x^{-1} \odot \mathbf{1} \odot \phi_{x'}^{-1}\big) \circ b \circ \big(\phi_x \odot \mathbf{1} \odot \phi_{x'}\big)$$

where

$$b : T^cS\mathcal{A} \odot SM \odot T^cS\mathcal{B} \to T^cS\mathcal{A} \odot SM \odot T^cS\mathcal{B}$$

is the differential of the bar construction of the $\mathcal{A}$-$\mathcal{B}$-bipolydule M.

Remark 6.1.4.3. Let $f : \mathcal{A} \to \mathcal{B}$ be an A$_\infty$-functor. Suppose that the twisting elements x and x' are compatible with f (see 6.1.3). Let

$$y : \mathcal{B} \to \mathcal{C}_\infty \mathcal{B}$$

be the Yoneda A$_\infty$-functor which will be defined in Section 7.1.0.1 By Corollary 5.3.0.3, the two compositions of A$_\infty$-functors

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{y} \mathcal{C}_\infty \quad \text{and} \quad \mathcal{A}_x \xrightarrow{f_x} \mathcal{B}_{x'} \xrightarrow{y} \mathcal{C}_\infty \mathcal{B}_{x'}$$

are given by a $\mathcal{A}$-$\mathcal{B}$-bipolydule M and a $\mathcal{A}_x$-$\mathcal{B}_{x'}$-bipolydule N.

We verify that we have

$${}_xM_{x'} = N.$$

6.2 The topological case

Let $\mathbb{A}$ be a set and let $\mathcal{A}$ be an A$_\infty$-category over $\mathbb{A}$. We are dealing here with the twisting of $\mathcal{A}$ by a morphism $x : e \to \mathcal{A}$ that is not tensorially nilpotent. The left-hand sum in the Maurer-Cartan equation (see 6.1.1.2)

$$\sum_{i \geq 1} m_i(x^{\odot i}) = 0$$

applied to $\mathbf{I}_A$ is no longer finite, but the equation still makes sense: if $\mathcal{A}$ is equipped with a topology, we interpret the equation above as the convergence of the series to zero. We show, using an algebraic trick, that the formulas providing twisted structures in the case where x is a tensorially nilpotent twisting element also give twisted structures in the case where $\mathcal{A}$ is topological and x satisfies the Maurer-Cartan equation.

6.2.1 Definitions

Terminology of topological objects

Let $(\mathsf{M}, \otimes, e)$ be a monoidal abelian $\mathbb{K}$-category. A *topology* on an object $V \in \mathsf{M}$ is a decreasing filtration

$$V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_i \supset \cdots$$

(see [Bou61, Chap. III §2 n°5]). A topology is *separated* if $\cap_{i\in\mathbf{N}} V_i = 0$. We will then say that the sub-objects V_i, $i \geq 1$, are a system of *neighborhoods* of 0. A *topological* object in M is an object M with a topology. topology. Its *completion* is the limit

$$\widehat{V} = \lim_{i\geq 0} V/V_i.$$

An object V is complete if $V = \widehat{V}$. Let V and V' be two topological objects. A *morphism* $f : V \to V'$ is a continuous morphism. It is *contractant* if it satisfies

$$f(V_i) \subset V_i', \quad i \geq 1.$$

The neutral element e for the tensor product is equipped with the discrete topology. The tensor product $V \otimes V'$ is topological for the system of neighborhoods

$$\big(V \otimes V'\big)_i = \sum_{i_1+i_2\geq i} V_{i_1} \otimes V_{i_2}, \quad i \geq 0.$$

The category of topological objects in M, equipped with a topological tensor product and a neutral object e, forms a monoidal category. The *complete tensor product* $V \widehat{\otimes} V'$ is defined as the limit

$$V\widehat{\otimes}V' = \lim_{i\geq 0} (V \otimes V')/(V \otimes V')_i.$$

The category of complete objects in M, equipped with the complete tensor product and the neutral element e is also a monoidal category.

Topological A_∞-structures

Let C be a base category (see 1.1.1).

Definition 6.2.1.1. An A_∞-algebra A in C is *topological* if A is equipped with a separated topology and if the multiplications $m_i : A^{\otimes i} \to A$, $i \geq 1$, are continuous contracting morphisms. Let A and A' be topological A_∞-algebras. A *topological* A_∞-morphism $f : A \to A'$ is an A_∞-morphism such that the morphisms f_i, $i \geq 1$, are continuous contracting morphisms. We define homotopies between A_∞-morphisms in a similar manner.

Let C' be a Grothendieck category equipped with a right action of the monoidal category C. This action extends to the category of topological objects of C' and C.

Definition 6.2.1.2. A *topological* $\mathcal{A}$-polydule in C' is a separated topological object M in C' equipped with a $\mathcal{A}$-polydule structure with multiplications m_i^M, $i \geq 1$, being continuous contracting morphisms. We define in a similar manner A_∞-morphisms and homotopies between A_∞-morphisms.

6.2.2 Twisting elements

Definition 6.2.2.1. Let A be a topological A_∞-algebra. A graded morphism $x : e \to A$ of degree $+1$ is a *(topological) twisting element* if its image is in the neighborhood A_1 and if the sum

$$\sum_{i\geq 1} m_i(x^{\otimes i})$$

converges to 0.

Remark 6.2.2.2. This sum converges to a well-defined limit because the topology of A is separated, the image of x is in A_1, and the multiplications m_1, $i \geq 1$, are contracting.

6.2.3 Local algebras

Let $\mathcal{R}$ be the category of $\mathbb{K}$-algebras that are local commutative rings R with residue field $\mathbb{K}$ and whose maximal ideal $\mathfrak{m}$ is nilpotent. Let R be an object of $\mathcal{R}$. We denote by $\mathcal{E}$ the category of modules over R. Let $\mathbb{O}$, $\mathbb{O}'$ and $\mathbb{O}''$ be three sets. We denote by $\mathsf{C}^R(\mathbb{O}, \mathbb{O}')$ the category of functors

$$\mathbb{O}'^{op} \times \mathbb{O} \to \mathcal{E}$$

and $\mathsf{C}^R(\mathbb{O}')$ the category $\mathsf{C}^R(\{*\}, \mathbb{O}')$. If M and N are objects in $\mathsf{C}^R(\mathbb{O}, \mathbb{O}')$ and $\mathsf{C}^R(\mathbb{O}', \mathbb{O}'')$, we denote by $\odot_R$ the tensor product

$$\Big(M \odot_R N\Big)(o'', o) = \bigoplus_{o' \in \mathbb{O}'} M(o', o) \otimes_R N(o'', o').$$

Definition 6.2.3.1. Let $\mathbb{A}$ be a set. An R-A_∞-*category* is an object M of $\mathsf{C}^R(\mathbb{A}, \mathbb{A})$, equipped with morphisms

$$m_i : M^{\odot_R i} \to M, \quad i \geq 1,$$

satisfying the equation $(*_n)$, $n \geq 1$, in the Definition 1.2.1.1. The R-A_∞-*functors* are defined as in 5.1.2.5

Let M and M' be objects in $\mathsf{C}(\mathbb{A}, \mathbb{A})$ and i an integer ≥ 1. Let

$$\varphi : M^{\odot i} \to M'$$

be a graded morphism. Let

$$\varphi^R : (M \otimes_{\mathbb{K}} \mathfrak{m})^{\odot_R i} \to M' \otimes_{\mathbb{K}} \mathfrak{m}$$

be a morphism in $\mathsf{C}^R(\mathbb{A}, \mathbb{A})$ defined by the composition

$$\varphi \otimes \mu^{(i)} : (M \otimes_{\mathbb{K}} \mathfrak{m})^{\odot_R i} \simeq (M^{\odot i}) \otimes_{\mathbb{K}} (\mathfrak{m})^{\otimes_R i} \to M' \otimes_{\mathbb{K}} \mathfrak{m}.$$

Notice that, since $\mathfrak{m}$ is nilpotent, there exists an integer N_0 such that $\mathfrak{m}^{N_0} = 0$. Therefore, the morphism φ^R is zero whenever $i \geq N_0$.

Remark 6.2.3.2. Let $\mathcal{A}$ be an object in $\mathsf{C}(\mathbb{A}, \mathbb{A})$ and

$$m_i : \mathcal{A}^{\odot i} \to \mathcal{A}, \quad i \geq 1,$$

be graded morphisms of degree $2 - i$. They satisfy the morphisms m_i, $i \geq 1$, defining an A_∞-category structure on $\mathcal{A}$ if and only if, for all $R \in \mathcal{R}$, the morphisms m_i^R, $i \geq 1$, define an R-A_∞-category structure on $\mathcal{A} \otimes_{\mathbb{K}} \mathfrak{m}$.

Let $\mathcal{A}$ be an A_∞-category and R an object of $\mathcal{R}$. We denote by $\mathcal{A}^R$ the R-A_∞-category $\mathcal{A} \otimes_{\mathbb{K}} \mathfrak{m}$ on $\mathbb{A}$ associated to $\mathcal{A}$.

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets and $\mathcal{A}$ and $\mathcal{B}$ be two A_∞-categories on $\mathbb{A}$ and $\mathbb{B}$. They satisfy the graded morphisms f_i, $i \geq 1$, of degree $1 - i$, defining an A_∞-functor

$$f : \mathcal{A} \to \mathcal{B}$$

if and only if, for all $R \in \mathcal{R}$, the morphisms f_i^R, $i \geq 1$, define an R-A$_\infty$-functor

$$f^R : \mathcal{A}^R \to \mathcal{B}^R.$$

Note that the morphisms m_i^R and f_i^R are zero as soon as i exceeds the nilpotency degree of the maximal ideal of R.

Bar construction B^R

Let R be an object in $\mathcal{R}$. The Lemma 1.1.2.2 remains valid in the category $\mathsf{C}^R(\mathbb{A}, \mathbb{A})$. In particular the bar construction defines a fully faithful functor

$$B^R : \mathsf{Alg}_\infty^R \to \mathsf{Cogc}^R,$$

where Alg_∞^R and Cogc^R are the categories Alg_∞ and Cogc in $\mathsf{C}^R(\mathbb{A}, \mathbb{A})$.

Reminder on completion Let R be an object in $\mathcal{R}$. Let V and W be two $\mathbb{A}$-$\mathbb{A}$-R-bimodules. We equip the *reduced tensor R-coalgebra*

$$\overline{T^c}V = \bigoplus_{i \geq 1} V^{\odot_R i}$$

with the *canonical topology* whose base neighborhood of 0 is

$$\bigoplus_{i \geq n} V^{\odot_R i}, \quad n \geq 1.$$

The coproduct is a continuous map for this topology. Recall that $T^c V$ denotes the co-augmented coalgebra $(\overline{T^c}V)^+$. We equip it with neighborhoods defined in the same way.

Remark 6.2.3.3. A morphism in $\mathsf{C}^R(\mathbb{A}, \mathbb{A})$

$$T^c V \to T^c W \quad \left(\text{resp.} \overline{T^c}V \to \overline{T^c}W\right)$$

is continuous if and only if the matrix of components

$$\bigoplus_{j \geq 0} V^{\odot_R j} \to \bigoplus_{i \geq 0} W^{\odot_R i} \quad \left(\text{resp.} \bigoplus_{j \geq 1} V^{\odot_R j} \to \bigoplus_{i \geq 1} W^{\odot_R i}\right)$$

has a finite number of nonzero components in each row. In particular, a coalgebra morphism f (resp. a (f', f'')-coderivation h, where f' and f'' are coalgebra morphisms)

$$\overline{T^c}V \to \overline{T^c}W$$

is continuous if and only if the morphisms f_i, $i \geq 0$, (resp. the morphisms f_i', f_i'' and h_i, $i \geq 0$,) are almost all zero.

The *reduced completed tensor R-coalgebra* $\widehat{T^c}V$ is the completion of $\overline{T^c}V$. Its underlying topological space is given by

$$\prod_{i \geq 1} V^{\odot_R i}.$$

Each continuous morphism $\varphi : \overline{T^c}V \to \overline{T^c}W$ in $\mathsf{C}^R(\mathbb{A},\mathbb{A})$ induces a morphism

$$\widehat{f} : \widehat{T^c}V \to \widehat{T^c}W.$$

The *completed co-augmented tensor coalgebra* $\widehat{T^{c+}}V$ is the co-augmentation of $\widehat{T^c}V$.

Lemma 6.2.3.4. Let V be an object in $\mathcal{G}r\mathsf{C}^R(\mathbb{A},\mathbb{A})$ and C a topological graded coalgebra in $\mathsf{C}^R(\mathbb{A},\mathbb{A})$. Let f' and f'' be two continuous morphisms of coalgebras

$$C \to \widehat{T^{c+}}V.$$

A continuous co-unital morphism from completed coalgebras (resp. a (f', f'')-coderivation) $C \to \widehat{T^{c+}}V$ is determined by its composition with the projection $\widehat{T^{c+}}V \to V$. □

6.2.4 Torsion of A_∞-categories

Torsion of the differential of $B\mathcal{A}^R$

Let $\mathbb{A}$ be a set. Let $\mathcal{A}$ be a topological A_∞-category over $\mathbb{A}$, i.e. a topological A_∞-algebra in $\mathsf{C}(\mathbb{A},\mathbb{A})$. Let $x : e \to \mathcal{A}$ be a twisting (topological) element of $\mathcal{A}$.

Let R be an object of $\mathcal{R}$. Let N_0 be the index of nilpotence of its maximal ideal $\mathfrak{m}$. Let $\mathcal{A}^R$ be the R-A_∞-category over $\mathbb{A}$ associated with $\mathcal{A}$. Let $\overline{T^c}S\mathcal{A}^R$ be the reduced tensorial R-coalgebra, and $\widehat{T^c} + S\mathcal{A}^R$ be the co-augmented R-coalgebra associated with its completion. The differential of the bar construction $B^R\mathcal{A}^R$

$$b^R : \overline{T^c}S\mathcal{A}^R \to \overline{T^c}S\mathcal{A}^R$$

is continuous because the morphisms m_i^R are zero for $i \geq N_0$. Let $\widehat{b}^R$ be the differential of $\widehat{T^c}{+}S\mathcal{A}^R$ induced by b^R. Let

$$x^R : e^R \to \mathcal{A}^R$$

be the morphism induced by x and

$$g : e \oplus \widehat{T^c}S\mathcal{A}^R = \widehat{T^{c+}}S\mathcal{A}^{R+} \to S\mathcal{A}^R.$$

be the morphism whose components are the morphisms x^R and the projection p_1 onto $S\mathcal{A}^R$. Let the morphism of $\mathbb{A}$-$\mathbb{A}$-R-bimodules

$$\phi_x^R : \widehat{T^{c+}}\mathcal{A}^R \to \widehat{T^{c+}}\mathcal{A}^R$$

whose composition with the projection onto $(S\mathcal{A}^R)^{\odot n}$ is equal to

$$g^{\odot n} \circ \Delta^{(n)} : \widehat{T^{c+}}S\mathcal{A}^R \to (S\mathcal{A}^R)^{\odot n}$$

if $n \geq 1$ and $\mathbf{1}_e$ otherwise. Like the morphism ϕ_x from Section 6.1.2, the morphism ϕ_x^R is a continuous co-unital (non-coaugmented) automorphism of graded coalgebras, and the matrix of its coefficients

$$\prod_{j\geq 0}(S\mathcal{A}^R)^{\odot_R j} \to \prod_{i\geq 0}(S\mathcal{A}^R)^{\odot_R i}$$

is lower triangular, with its diagonal being that of the identity. Consider the composition

$$D_x^R = (\phi_x^R)^{-1} \circ \widehat{b^{R+}} \circ \phi_x^R.$$

Since x is a twisting element, we have

$$\sum_{1\leq i\leq N_0} \widehat{b}_i^R\big((x^R)^{\odot_R i}\big) = 0.$$

Note that the lack of tensorial nilpotence is compensated by the vanishing of morphisms b_i^R for $i \geq N_0$. As in Section 6.1.2, the composition $D_x^R \circ \varepsilon$ is null. Let b_x^R be the morphism given by the right vertical arrow in the diagram of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & e \arrow[r, "\varepsilon"] \arrow[d, "0"'] & \widehat{T^{c+}}S\mathcal{A}^R \arrow[r] \arrow[d, "D_x^R"'] & \widehat{T^c}S\mathcal{A}^R \arrow[r] \arrow[d, "b_x^R"'] & 0 \\
0 \arrow[r] & e \arrow[r, "\varepsilon"] & \widehat{T^{c+}}S\mathcal{A}^R \arrow[r] & \widehat{T^c}S\mathcal{A}^R \arrow[r] & 0.
\end{tikzcd}$$

It is a differential for the coalgebra $\widehat{T^c}S\mathcal{A}^R$.

Lemma 6.2.4.1. The sub-coalgebra $T^cS\mathcal{A}^R$ of $\widehat{T^c}S\mathcal{A}^R$ is stable under the differential b_x^R. The composite $p_1^R \circ b_x^R$ restricted to $(S\mathcal{A}^R)^{\odot i}$ is equal to the sum

$$\sum_l \sum b_{l+m}^R((sx)^{\odot l_0} \odot \mathbf{1}_{S\mathcal{A}^R} \odot (sx)^{\odot l_1} \odot \ldots \odot \mathbf{1}_{S\mathcal{A}^R} \odot (sx)^{\odot l_{i-1}} \odot \mathbf{1}_{S\mathcal{A}^R} \odot (sx)^{\odot l_i}),$$

where $l_0 + \ldots + l_i = l$.

Proof. Identical to the one in Lemma 6.1.2.1 □

A_∞-category twisted by x

Let $\mathbb{A}$ be a set, $\mathcal{A}$ a topological A_∞-category over $\mathbb{A}$, and $x : e \to \mathcal{A}$ a twisting element. Consider the morphisms

$$m_i^x : \mathcal{A}^{\odot i} \to \mathcal{A}, \quad i \geq 1,$$

defined by the sum

$$\sum_l \sum (-1)^s m_{l+i}^{\mathcal{A}}(x^{\odot l_0} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_1} \odot \ldots \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_{i-1}} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_i}),$$

where the exponent of the sign is $s = \sum_{1\leq t\leq i} t \times l_t$. Note that these sums converge to well-defined limits because $\mathcal{A}$ is topologically separated, the image of x is in the neighborhood $\mathcal{A}_1$, and the compositions m_i, $i \geq 1$, are continuous contracting morphisms.

Lemma 6.2.4.2. The morphisms m_i^x, $i \geq 1$, define a structure of an A_∞-category on the $\mathbb{A}$-$\mathbb{A}$-bimodule underlying $\mathcal{A}$.

Proof. The lemma remains valid if, for any object $R \in \mathcal{R}$, the morphisms $(m_i^x)^R$, $i \geq 1$, define a structure of an R-A_∞-category on the R-$\mathbb{A}$-$\mathbb{A}$-bimodule underlying $\mathcal{A}^R$.

Let R be an object of $\mathcal{R}$. We verify that the morphism b_x^R from Lemma 6.2.4.1 is the coderivation

$$T^c(S\mathcal{A}^R) \to T^c(S\mathcal{A}^R)$$

constructed from $(m_i^x)^R$, $i \geq 1$. Since it is a differential, we have the result. □

Definition 6.2.4.3. The *(topological) twisted* A_∞*-category* $\mathcal{A}_x$ is the $\mathbb{A}$-$\mathbb{A}$-bimodule $\mathcal{A}_x = \mathcal{A}$ equipped with compositions

$$m_i^x : \mathcal{A}_x^{\odot i} \to \mathcal{A}_x, \quad i \geq 1,$$

defined below.

6.2.5 Torsion of A_∞-functors

Let $\mathbb{A}$ and $\mathbb{B}$ two sets, $\mathcal{A}$ and $\mathcal{B}$ be two topological A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$, and x and x' are twisting elements in $\mathcal{A}$ and $\mathcal{B}$ such that for every $A \in \mathbb{A}$,

$$\sum_{i\geq 1} f_i(x^{\odot i})(\mathbf{I}_A) = \mathbf{I}_{\dot{f}A}.$$

Note that the left sum converges to a well-defined limit since $\mathcal{B}'$ is topologically separated, the image of x is in the neighborhood $\mathcal{A}_1$ and because the morphisms f_i, $i \geq 1$, are contracting. The above equality expresses the compatibility of x and x' with respect to f (see 6.1.3). Let's revist the notations $\mathcal{B}'$, $\mathcal{B}'_{x'}$ from Section 6.1.3. Consider the morphisms

$$f_i^x : \mathcal{A}^{\odot i} \to \mathcal{B}', \quad i \geq 1,$$

defined by the (convergent) sum

$$\sum_l \sum (-1)^s f^{\mathcal{A}}_{l+i}(x^{\odot l_0} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_1} \odot \ldots \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_{i-1}} \odot \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_i}),$$

where the exponent of the sign is $s = \sum_{1\leq t\leq i} t \times l_t$.

Lemma 6.2.5.1. The morphisms f_i^x, $i \geq 1$, define an A_∞-functor

$$(\dot{f}, f_x) : \mathcal{A}_x \to \mathcal{B}_{x'}.$$

Proof. We will show that, for every object $R \in \mathcal{R}$, the morphisms f_i^R, $i \geq 1$, define an A_∞-functor

$$f_x^R : \mathcal{A}_x^R \to \mathcal{B}'^R_{x'},$$

or equivalently, a differential graded morphism of coalgebras

$$F_x^R : B^R\mathcal{A}_x^R \to B^R\mathcal{B}'^R_{x'}.$$

Let $R \in \mathcal{R}$. Due to the compatibility of x and x' with f, the graded differential morphism of complete counital coalgebras

$$G^R = (\phi^R_{x'})^{-1} \circ \widehat{F}^+ \circ \phi^R_x : \widehat{T^c}^+ S\mathcal{A}^R_x \to \widehat{T^c}^+ S\mathcal{B}'^R_{x'}$$

is coaugmented. Therefore, it induces a differential graded morphism

$$F_x : (\widehat{T^c} S\mathcal{A}^R_x, \widehat{b}^R_x) \to (\widehat{T^c} S\mathcal{B}'^R_{x'}, \widehat{b'}^R_{x'}).$$

Let $i \geq 1$. We show, similarly to the proof of Lemma 6.2.4.1, that the restriction of F_x to the subobject $(S\mathcal{A}^R_x)^{\odot i}$ is equal to the sum

$$\sum_l \sum F^R_{l+m}((sx)^{\odot l_0} \odot sa_1 \odot (sx)^{\odot l_1} \odot \ldots \odot sa_{i-1} \odot (sx)^{\odot l_{i-1}} \odot sa_i \odot (sx)^{\odot l_i}),$$

where $l_0 + \ldots + l_i = l$. This sum is finite because the morphisms F_i^R are zero if i exceeds the nilpotence degree of the maximal ideal of R. We thus obtain a morphism of coalgebras

$$F_x^R : (\overline{T^c} S\mathcal{A}^R_x, b^R_x) \to (\overline{T^c} S\mathcal{B}'^R_{x'}, b'^R_{x'})$$

that is differential graded. So, we have the result. $\square$

Definition 6.2.5.2. The *twisted* A_∞*-functors*

$$(\dot{f}, f^x) : \mathcal{A}_x \to \mathcal{B}_{x'}$$

are given by the morphisms f_i^x, $i \geq 1$, defined above.

The proposition (6.1.3.4) clearly remains valid in the topological case.

6.2.6 Torsion of $\mathcal{A}$-$\mathcal{B}$-bipolydules

The details are omitted as they are similar to the last two sections.

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets, $\mathcal{A}$ and $\mathcal{B}$ be two topological A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$, and M be a topological $\mathcal{A}$-$\mathcal{B}$-bipolydule. Let x and x' be twisting elements in $\mathcal{A}$ and $\mathcal{B}$.

Definition 6.2.6.1. A $\mathcal{A}_x$-$\mathcal{B}_{x'}$*-bipolydule* ${}_xM_{x'}$ has multiplications

$$m_{i,j}^{x,x'} : \mathcal{A}_x^{\odot i} \odot {}_xM_{x'} \odot \mathcal{B}_{x'} \to {}_xM_{x'}, \quad i, j \geq 0,$$

defined by the (convergent) sum

$$\sum_{l,k\geq 0} \sum (-1)^s m_{i+l,j+k}(x^{\odot l_0} \odot \mathbf{1}_{\mathcal{A}} \ldots \mathbf{1}_{\mathcal{A}} \odot x^{\odot l_i} \odot \mathbf{1}_M \odot {x'}^{\odot k_o} \odot \mathbf{1}_{\mathcal{B}} \ldots \mathbf{1}_{\mathcal{B}} \odot {x'}^{\odot k_j}),$$

where the exponent of the sign is

$$s = \Big(\sum_{1\leq t\leq i} t \times l_t \Big) + \Big(\sum_{1\leq t\leq j} (j+t) \times l_t \Big)$$

Chapter 7

The Yoneda A_∞-functor and twisted objects

Introduction

Let $\mathbb{A}$ be a set and $\mathcal{A}$ a strictly unital A_∞-category over $\mathbb{A}$. Let $\mathcal{G}r(H^*\mathcal{A})$ denote the category of graded $H^*\mathcal{A}$-modules, with graded morphisms. In this section, we lift the Yoneda functor

$$H^*\mathcal{A} \to \mathcal{G}r(H^*\mathcal{A}), \quad A \mapsto (H^*\mathcal{A})(-, A),$$

into an A_∞-functor

$$y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}, \quad A \mapsto \mathcal{A}(-, A).$$

We show the main result of this chapter (7.1.0.4): *The A_∞-functor y factorizes as*

$$\mathcal{A} \xrightarrow{y'} \mathsf{tw}\mathcal{A} \xrightarrow{y''} \mathcal{C}_\infty\mathcal{A}$$

where $\mathsf{tw}\mathcal{A}$ is the A_∞-category of twisted objects, y' is a strict and fully faithful A_∞-functor and y'' induces an equivalence

$$H^0\mathsf{tw}\mathcal{A} \xrightarrow{\sim} \mathsf{tria}\,\mathcal{A} \subset \mathcal{D}_\infty\mathcal{A}.$$

The construction of twisted objects in the case where $\mathcal{A}$ is differential graded is due to A. I. Bondal and M. M. Kapranov [BK91], with a generalization to A_∞-categories by M. Kontsevich [Kon95] Recently, K. Fukaya independently constructed the Yoneda A_∞-functor [Fuk01b].

Chapter Plan

In Section 7.1, we define the Yoneda A_∞-functor and state the main theorem (7.1.0.4). The rest of the chapter (except Section 7.5) is dedicated to proving this theorem. In Section 7.2, we construct the A_∞-category $\mathsf{tw}\mathcal{A}$ of twisted objects. The compositions in the A_∞-category $\mathsf{tw}\mathcal{A}$ are obtained through torsion (see Chapter 6). We then demonstrate that the A_∞-category $\mathsf{tw}\mathcal{A}$ possesses a universal property, from which we deduce the existence of the factorization $y'' \circ y'$ of y. In Section 7.3, we explicitly construct the A_∞-functor y''. In Section 7.4, we show that the Yoneda A_∞-functor y induces quasi-isomorphisms between morphism spaces, leading to the equivalence

$$H^0\mathsf{tw}\mathcal{A} \simeq \mathsf{tria}\,\mathcal{A} \subset \mathcal{D}_\infty\mathcal{A}.$$

In Section 7.5, we demonstrate that every homologically unital A_∞-category $\mathcal{A}$ has a *strictly unital differential graded model*, which means a homologically unital A_∞-quasi-isomorphism $f\colon \mathcal{A} \to \mathcal{A}'$ to a strictly unital differential graded category.

In Section 7.6, we show that any algebraic triangulated category generated by a set of objects is A_∞-pre-triangulated, meaning it is equivalent to $H^0\mathsf{tw}\mathcal{A}$ for a certain A_∞-category $\mathcal{A}$.

7.1 The Yoneda embedding

As $\mathcal{A}$ is an A_∞-category, the $\mathbb{A}$-$\mathbb{A}$-bimodule $\mathcal{A}$, equipped with morphisms $m_{i,j} = m^{\mathcal{A}}_{i+1+j}$, $i, j \geq 0$, is an $\mathcal{A}$-$\mathcal{A}$-bipolydule. By Remark 5.3.0.5, we have an A_∞-functor:

$$y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A},$$

whose underlying map:

$$\dot{y} : \mathbb{A} \to \mathcal{C}_\infty\mathcal{A}$$

sends an object $A \in \mathcal{A}$ to the $\mathcal{A}$-*polydule*

$$A^\wedge = \mathcal{A}(-, A).$$

For all $i \geq 1$, the graded morphisms

$$y_i : \mathcal{A}^{\odot i} \to {}_{\dot{f}}\big(\mathcal{C}_\infty\mathcal{A}\big)_{\dot{f}},$$

send an element $x \in (\mathcal{A}^{\odot i})(A, A')$ to the sequence of morphisms of graded $\mathbb{A}$-modules

$$\begin{array}{rcll} \mathcal{A}(-, A) \odot \mathcal{A}^{\odot j-1} & \to & \mathcal{A}(-, A'), & \qquad j \geq 1. \\ x' \odot x'' & \mapsto & (-1)^{|x|+1} m_{i+1+j}(x' \odot x \odot x'') & \end{array}$$

Definition 7.1.0.1. The A_∞-*Yoneda functor* is the A_∞-functor $y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$.

Definition 7.1.0.2. A strict A_∞-functor f is *fully faithful* if

$$f_1 : \mathcal{A} \to {}_{\dot{f}}\mathcal{B}_{\dot{f}}$$

is an isomorphism of complexes.

Definition 7.1.0.3. Let $\mathcal{T}$ be a triangulated category and $\mathbb{T}'$ be a subset of the set $\mathbb{T}$ of objects of $\mathcal{T}$. Denote by $\mathsf{tria}\,\mathbb{T}'$ the *smallest triangulated subcategory* of $\mathcal{T}$ which contains the objects of $\mathbb{T}'$. It is stable under finite sums. Let $\mathcal{A}$ be a strictly unital A_∞-category and $\mathcal{D}_\infty\mathcal{A}$ its derived category (see 4.1.2). Denote by $\mathsf{tria}\,\mathcal{A}$ the smallest triangulated subcategory of $\mathcal{D}_\infty\mathcal{A}$ which contains all the $\mathcal{A}$-polydues $A^\wedge$, $A \in \mathsf{Obj}\,\mathcal{A}$.

In this chapter, we will prove the following statement of M. Kontsevich [Kon95], [Kon98]:

Theorem 7.1.0.4 (see also K. Fukaya [Fuk01b])**.** Let $\mathcal{A}$ be an A_∞-category with strict identities. There exists an A_∞-category $\mathsf{tw}\mathcal{A}$ and a factorization of the Yoneda A_∞-functor

$$\mathcal{A} \xrightarrow{y'} \mathsf{tw}\mathcal{A} \xrightarrow{y''} \mathcal{C}_\infty\mathcal{A}$$

such that the Yoneda A_∞-functor y' is strict and fully faithful and the A_∞-functor y'' induces an equivalence

$$H^0\mathsf{tw}\mathcal{A} \simeq \mathsf{tria}\,\mathcal{A} \subset \mathcal{D}_\infty\mathcal{A}.$$

Proof. See the following three sections. $\square$

7.2 The A_∞-category of twisted objects

Let Λ be an associative unital (not graded) algebra. We denote by $\mathcal{C}^b(\mathsf{free}\,\Lambda)$ the *subcategory* of $\mathcal{C}\Lambda$ consisting of bounded complexes of free and finite rank Λ-modules. The image $\mathcal{D}^b(\mathsf{free}\,\Lambda)$ of the category $\mathcal{C}^b(\mathsf{free}\,\Lambda)$ under the functor

$$\mathcal{C}\Lambda \to \mathcal{D}\Lambda$$

is equivalent to the category $\mathsf{tria}\,\Lambda$. The objects of $\mathcal{C}^b(\mathsf{free}\,\Lambda)$ are fibrants and cofibrants in the category of complexes $\mathcal{C}\Lambda$. If M and M' are objects of $\mathcal{D}^b(\mathsf{free}\,\Lambda)$, the morphisms $M \to M'$ in $\mathsf{tria}\,\Lambda$ are in bijection with the homotopy classes of morphisms $M \to M'$ of $\mathsf{Mod}\,\Lambda$. This description of morphisms allows us to carry out calculations in $\mathsf{tria}\,\Lambda$. The purpose of this section is to generalize the construction

$$\Lambda \rightsquigarrow \mathcal{C}^b(\mathsf{free}\,\Lambda)$$

to A_∞-categories. Let $\mathcal{A}$ be an A_∞-category. The role of the category $\mathcal{C}^b(\mathsf{free}\,\Lambda)$ will be played by the A_∞-category $\mathsf{tw}\mathcal{A}$ of twisted objects. The equivalence between $\mathcal{D}^b(\mathsf{free}\,\Lambda)$ and $\mathsf{tria}\,\Lambda$ will be replaced by an equivalence

$$H^0\mathsf{tw}\mathcal{A} \xrightarrow{\sim} \mathsf{tria}\,\mathcal{A} \subset \mathcal{D}_\infty\mathcal{A}.$$

The construction $\mathcal{A} \rightsquigarrow \mathsf{tw}\mathcal{A}$ is the generalization to A_∞-categories [Kon95] of the construction due to A. I. Bondal and M. M. Kapranov [BK91] which associates to a differential graded category the category of its twisted objects (see 7.2.0.4).

To make the following construction more intuitive, we will start by reinterpreting the objects of $\mathcal{C}^b(\mathsf{free}\,\Lambda)$.

A bounded complex M of free and finite rank Λ-modules is given by its components

$$(M_r, M_{r+1}, \dots, M_{l-1}, M_l), \quad r \le l, \quad r, l \in \mathbf{Z},$$

where each M_i, $r \le i \le l$, is the iterated suspension of a Λ-module free of finite rank, and by a morphism of degree $+1$

$$\delta : \bigoplus_{r\le j\le l} M_j \to \bigoplus_{r\le i\le l} M_i$$

whose matrix is strictly lower triangular and such that $\delta \circ \delta = 0$.

Now suppose that Λ is a differential graded algebra. Iterated extensions in the category of complexes, equipped with the exact structure given by sequences of complexes that split as sequences of graded Λ-modules, are described as follows. Let M_i, for $r \le i \le l$, be objects in $\mathsf{Mod}\,\Lambda$ which are finite sums of iterated suspensions of Λ. Denote by d the differential of the sum of the M_i, for $r \le i \le l$. In iterated extnesion of objects M_i, $r \le i \le l$, is given by a matrix of the same form as above which satisfies the Maurer-Cartan equation

$$d \circ \delta + \delta \circ d + \delta^2 = 0.$$

The differential of the iterated extension $M = \bigoplus_{r\le j\le l} M_j$ is the sum $d + \delta$.

Saturation by shifts of $\mathcal{A}$

Let $\mathbf{Z}\mathcal{A}$ be the A_∞-*category* whose objects are pairs (A, n), where A is an object of $\mathcal{A}$ and n is an integer. The morphism spaces are defined by

$$\mathbf{Z}\mathcal{A}((A,n),(B,m)) = S^{m-n}\mathcal{A}(A,B).$$

The compositions $m_i^{\mathbf{Z}\mathcal{A}}$, $i \geq 1$,

$$\begin{array}{c} \mathbf{Z}\mathcal{A}((A_{i-1},n_{i-1}),(A_i,n_i)) \otimes \ldots \otimes \mathbf{Z}\mathcal{A}((A_0,n_0),(A_1,n_1)) \\ \downarrow{\scriptstyle m_i^{\mathbf{Z}\mathcal{A}}} \\ \mathbf{Z}\mathcal{A}((A_0,n_0),(A_i,n_i)) \end{array}$$

are defined by

$$(-1)^{i(n_i-n_0)} s^{n_i-n_o} \circ m_i \circ \big((s^{n_i-n_{i-1}})^{-1} \odot \ldots \odot (s^{n_1-n_0})^{-1}\big)$$

(a calculation shows that these compositions define a A_∞-category).

Saturation by extensions of $\mathbf{Z}\mathcal{A}$

Definition 7.2.0.1. An *iterated extension* M of objects of $\mathbf{Z}\mathcal{A}$ is a sequence

$$(M_r, M_{r+1}, \ldots, M_{l-1}, M_l), \quad r \leq l, \quad r, l \in \mathbf{Z},$$

equipped with a matrix of coefficients in $\mathbf{Z}\mathcal{A}$ of degree $+1$

$$\delta^M : \bigoplus_{r\leq j\leq l} M_j \to \bigoplus_{r \leq i \leq l} M_i$$

which is strictly lower triangular and satisfies the Maurer-Cartan equation

$$\sum_{i\geq 1} m_i^{\mathbf{Z}\mathcal{A}}\big((\delta^M)^{\odot i}\big) = 0.$$

Here, the tensor product $\odot$ is the extension of the tensor product from $\mathsf{C}(\mathbb{A},\mathbb{A})$ to the space of matrices with coefficients in $\mathbf{Z}\mathcal{A}$. The integer $l-n+1$ is called the *height* of the extension. An iterated extension M is *degenerate* or *split* if $\delta^M = 0$. Degenerate iterated extensions can be considered as formal sums of objects of $\mathbf{Z}\mathcal{A}$. We denote by $\mathbb{E}$ the set of iterated extensions of $\mathbf{Z}\mathcal{A}$.

Definition 7.2.0.2. Let M and M' be two iterated extensions of $\mathbf{Z}\mathcal{A}$. Denote by $\mathsf{Mat}^{\mathbf{Z}\mathcal{A}}(M, M')$ the graded space of matrices with coefficients in $\mathbf{Z}\mathcal{A}$

$$f : \bigoplus_{r\leq j\leq l} M_j \to \bigoplus_{r' \leq i \leq l'} M'_i.$$

The compositions $m_i^{\mathbf{Z}\mathcal{A}}$, $i \geq 1$, of $\mathbf{Z}\mathcal{A}$ clearly extend to compositions of matrices with coefficients in $\mathbf{Z}\mathcal{A}$. Denote by $\mathcal{E}_\mathcal{A}$ the A_∞-*category* whose objects are iterated extensions of objects of $\mathbf{Z}\mathcal{A}$ and whose morphism spaces are

$$\mathsf{Hom}_{\mathcal{E}_\mathcal{A}}(M, M') = \mathsf{Mat}^{\mathbf{Z}\mathcal{A}}(M, M').$$

We clearly have a sequence of inclusions of A_∞-categories

$$\mathcal{A} \subset \mathbf{Z}\mathcal{A} \subset \mathcal{E}_\mathcal{A}.$$

The niltpotent twisting element of the A_∞-category $\mathcal{E}_\mathcal{A}$

We recall (5.1.1) that $\mathbf{I}_M$ is the generator of the space $e_\mathbb{E}(M,M)$. Let

$$x : e_\mathbb{E} \to \mathcal{E}_\mathcal{A}$$

be the morphism of $\mathbb{E}$-$\mathbb{E}$-bimodules which sends $\mathbf{I}_M$, $M \in \mathbb{E}$, to

$$\delta^M \in \mathsf{Mat}^{\mathbf{Z}\mathcal{A}}(M,M).$$

The morphism x is of degree $+1$. It satisfies the condition of tensorial nilpotence (6.1.1.2) because the matrices δ^M are strictly lower triangular. Since the morphisms δ^M, $M \in \mathbb{E}$, satisfy the Maurer-Cartan equation, the morphism x is a tensorially nilpotent twisting element.

The category $\mathsf{tw}\mathcal{A}$

Definition 7.2.0.3. A *twisted object* is an iterated extension of objects of $\mathbf{Z}\mathcal{A}$. Denote by $\mathbb{TW}\mathcal{A}$ the set of twisted objects. It is equal to the set $\mathbb{E}$. The *category* $\mathsf{tw}\mathcal{A}$ of twisted objects is the twisted category $\left(\mathcal{E}_\mathcal{A}\right)_x$ (see 6.1.2), where x is the twisting element above.

If M and M' are twisted objects, the space of morphisms $M \to M'$ is given by

$$\mathsf{Hom}_{\mathsf{tw}\mathcal{A}}(M,M') = \mathsf{Mat}^{\mathbf{Z}\mathcal{A}}(M,M').$$

Note that on the sub-A_∞-category consisting of degenerate extensions, the twisted compositions $m_i^{\mathcal{E}_x} = m_i^{\mathsf{tw}\mathcal{A}}$, $i \geq 1$, are equal to the $\mathcal{E}$-compositions $m_i^{\mathcal{E}}$, $i \geq 1$. Let $\mathbb{E}_1$ be the set of (necessarily degenerate) extensions of height 1, and let

$$\dot{y}' : \mathbb{A} \to \mathbb{E}_1,$$

be the map that sends A to the degenerate extension of height 1 whose underlying sequence is the 1-tuple $((A,0))$. This is a bijetion and we have an isomorphism

$$y_1' : \mathcal{A} \xrightarrow{\sim} {}_{\dot{y}'}\mathsf{Mat}^{\mathbf{Z}\mathcal{A}}{}_{\dot{y}'} = {}_{\dot{y}'}\mathsf{tw}\mathcal{A}_{\dot{y}'}$$

which clearly gives a strict and fully faithful A_∞-functor

$$y' : \mathcal{A} \to \mathsf{tw}\mathcal{A}.$$

The universal property of $\mathsf{tw}\mathcal{A}$

We are inspired by the article [BK91].

Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor. It clearly induces an A_∞-functor

$$f : \mathcal{E}_\mathcal{A} \to \mathcal{E}_\mathcal{B}$$

such that the twisting elements $x_\mathcal{A}$ and $x_\mathcal{B}$ of A_∞-categories $\mathcal{E}_\mathcal{A}$ and $\mathcal{E}_\mathcal{B}$ are compatible with f (see 6.1.3). We then obtain a twisted A_∞-functor (see 6.1.3)

$$\mathsf{tw}f : \mathsf{tw}\mathcal{A} \to \mathsf{tw}\mathcal{B}.$$

The construction which associates to an A_∞-category $\mathcal{A}$ the category of twisted objects $\mathsf{tw}\mathcal{A}$ is a functor

$$\mathsf{tw} : \mathsf{cat}_\infty \to \mathsf{cat}_\infty,$$

where cat_∞ is the category of small A_∞-categories. We will construct a morphism of functors

$$\mathsf{Tot} : \mathsf{tw} \circ \mathsf{tw} \to \mathsf{tw}.$$

Let $\mathcal{A}$ be a small A_∞-category. The strict A_∞-functor $\mathsf{Tot}(\mathcal{A})$ associates to an object N of $\mathsf{tw}\circ\mathsf{tw}\mathcal{A}$, given by a sequence of objects of $\mathsf{tw}\mathcal{A}$

$$(N_r, \dots, N_l), \quad r \leq l, \quad r, l \in \mathbf{Z},$$

and a matrix δ^N with coefficients in $\mathbf{Z}\mathsf{tw}\mathcal{A}$, the twisted object of $\mathcal{A}$ whose underlying sequence is the concatenation of sequences defining the N_i, $r \leq i \leq l$, and whose matrix

$$\delta^{\mathsf{Tot}} : \mathsf{Tot}(N) = \bigoplus (N_j)_k \to \mathsf{Tot}(N) = \bigoplus (N_i)_l$$

is constructed from the matrix δ^N by replacing the coefficients $\delta^N_{i,j}$ by the blocks given by the matrices δ^{N_i}. We verify that the morphisms of functors in cat_∞

$$\eta = y' : \mathbf{1}_{\mathsf{cat}_\infty} \to \mathsf{tw} \quad \text{and} \quad \mathsf{Tot} : \mathsf{tw} \circ \mathsf{tw} \to \mathsf{tw}$$

define a monad in the category of A_∞-categories in the sense of Quillen and Mac Lane [May72]. We recall that a tw-algebra $\mathcal{G}$ is an A_∞-category endowed with an A_∞-functor

$$\mathsf{tw}\mathcal{G} \to \mathcal{G}$$

that is compatible with the structure of a monad. The category $\mathsf{tw}\mathcal{A}$ is clearly the free tw-algebra on $\mathcal{A}$. In particular, the A_∞-functor $y' : \mathcal{A} \to \mathsf{tw}\mathcal{A}$ is universal among the A_∞-functors

$$\mathcal{A} \to \mathcal{G}$$

where $\mathcal{G}$ is an algebra over the monad.

Remark 7.2.0.4. If $\mathcal{G}$ is a differential graded category, $\mathsf{tw}\mathcal{G}$ is a differential graded category. The construction $\mathcal{G} \rightsquigarrow \mathsf{tw}\mathcal{G}$ corresponds to the construction of A. I. Bondal and M. M. Kapranov which associates to $\mathcal{G}$ the category $\text{Pr-Tr}^+\mathcal{G}$ of unilateral twisted objects [BK91, §4].

Existence of an A_∞-functor y''

Let $\mathcal{A}$ be a small A_∞-category. Let

$$\mathsf{tw}\mathcal{C}_\infty\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$$

be the strict A_∞-functor which associates to an iterated extension M the sum of the M_i, $r \leq i \leq l$, endowed with the differential $d + \delta_M$, where d is the differential of the sum of the M_i. This A_∞-functor defines a structure of tw-algebra on $\mathcal{C}_\infty\mathcal{A}$. In particular, the A_∞-functor

$$y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$$

factors as $y = y'' \circ y'$, where y'' is the A_∞-functor

$$\mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$$

given by the universal property of $\mathsf{tw}\mathcal{A}$.

7.3 The A_∞ functor $y'' : \mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$

In this section, we explicitly construct the A_∞-functor

$$y'' : \mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}.$$

By remark 5.3.0.6, the A_∞-functors

$$\mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$$

are in bijection with the strictly unital $\mathsf{tw}\mathcal{A}$-$\mathcal{A}$-bipolydules. The $\mathsf{tw}\mathcal{A}$-$\mathcal{A}$-bipolydule N'' associated to y'' is constructed by twisting (see Section 6.1.4) a $\mathcal{E}$-$\mathcal{A}$-bipolydule N. The A_∞-functor

$$f : \mathcal{E} \to \mathcal{C}_\infty\mathcal{A}$$

associated to N is the extension of the Yoneda A_∞-functor $y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$. We provide the explicit formulas for the A_∞-functors f and y''.

Construction of $f : \mathcal{E} \to \mathcal{C}_\infty\mathcal{A}$

We recall (7.2.0.2) that we have a sequence of inclusions of A_∞-categories

$$\mathcal{A} \subset \mathbf{Z}\mathcal{A} \subset \mathcal{E}$$

and that $y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$ denotes the Yoneda A_∞-functor (7.1.0.1). This last one extends to an A_∞-functor

$$\mathbf{Z}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}, \quad (A, n) \mapsto S^n(\dot{y}A) = S^n A^\wedge$$

which sends an element

$$x \in \mathbf{Z}\mathcal{A}((A_{i-1}, n_{i-1}), (A_i, n_i)) \otimes \ldots \otimes \mathbf{Z}\mathcal{A}((A_0, n_0), (A_1, n_1))$$

to the morhpism of $\mathcal{A}$-polydues $S^{n_0}A_0{}^\wedge \to S^{n_i}A_i{}^\wedge$ defined by the element of

$$\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(S^{n_0}A_0{}^\wedge, S^{n_i}A_i{}^\wedge) \simeq S^{n_i - n_0}\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(A_0{}^\wedge, A_i{}^\wedge)$$

given by

$$s^{n_i - n_o} \circ y_i \circ \big((s^{n_i - n_{i-1}})^{-1} \odot \ldots \odot (s^{n_1 - n_0})^{-1}\big)(x).$$

We also denote this A_∞-functor y. We now extend it to an A_∞-functor

$$\mathcal{E} \to \mathcal{C}_\infty\mathcal{A}.$$

We define a map

$$\dot{f} : \mathbb{E} \to \mathsf{Obj}\,\mathcal{C}_\infty\mathcal{A}$$

which sends an iterated extension M, given by a sequence M_i, $r \le i \le l$, and a matrix δ^M, to the $\mathbb{A}$-module which is the sum

$$\sum_{r \le i \le l} \dot{y}M_i.$$

Its structure as a $\mathcal{A}$-polydule is induced by that of Remark 5.1.2.9. Note that the matrix δ^M does not appear in the definition of the image of M. The morphisms $y_i : (\mathbf{Z}\mathcal{A})^{\odot i} \to \mathcal{C}_\infty\mathcal{A}$ clearly extend to morphisms

$$\left(\mathsf{Mat}^{\mathbf{Z}\mathcal{A}}\right)^{\odot i} \to \mathcal{C}_\infty\mathcal{A}.$$

This provides us with an A_∞-functor which we denote $f : \mathcal{E} \to \mathcal{C}_\infty\mathcal{A}$ and we have clearly the factorization $y = f \circ y'$. By Remark 5.3.0.6, the A_∞-functor f is given by a $\mathcal{E}$-$\mathcal{A}$-bipolydule N which, as an $\mathbb{E}$-$\mathbb{A}$-bimodule, is

$$(A, M) \mapsto \bigoplus_{r\leq i\leq l} S^{n_i}\mathcal{A}(A, A_i),$$

whee $M_i = (A_i, n_i)$, $r \leq i \leq l$. Let us denote

$$m^N_{i,j} : \mathcal{E}^{\odot i} \odot N \odot \mathcal{A}^{\odot j} \to N, \quad i, j \geq 0.$$

the multiplications of the $\mathcal{E}$-$\mathcal{A}$-bipolydule N. They are clearly induced by the extension to $\mathbf{Z}\mathcal{A}$, then to $\mathcal{E}$, of the compositions

$$m^{\mathcal{A}}_{i,j} = m^{\mathcal{A}}_{i+1+j} : \mathcal{A}^{\odot i} \odot \mathcal{A} \odot \mathcal{A}^{\odot j} \to \mathcal{A}, \quad i, j \geq 0.$$

The A_∞-functor $y'' : \mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$

We recall (7.2.0.2) that x denotes the (nilpotent) twisting element of $\mathcal{E}$. By section 6.1.4, we can twist N into a $\mathcal{E}_x$-$\mathcal{A}$-polydule ${}_xN = N''$. Since the A_∞-category $\mathsf{tw}\mathcal{A}$ is by definition the twisted A_∞-category $\mathcal{E}_x$, we obtain an $\mathsf{tw}\mathcal{A}$-$\mathcal{A}$-bipolydule N'' and by Remark (5.3.0.6), an A_∞-functor

$$y'' : \mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}.$$

Below, we provide the explicit formulas defining it. The $\mathbb{TW}\mathcal{A}$-$\mathbb{A}$-bimodule N'' is given by

$$(A, M) \mapsto \bigoplus_{r\leq i\leq l} S^{n_i}\mathcal{A}(A, A_i).$$

As $\mathbb{TW}\mathcal{A} = \mathbb{E}$, it is isomorphic as an $\mathbb{E}$-$\mathbb{A}$-bimodule to N. As a $\mathcal{E}_x$-$\mathcal{A}$-bipolydule, its multiplications $m^{N''}_{i,j}$, $i, j \geq 0$ are given (6.1.4.1) by the sum

$$\sum_{l,k\geq 0} \sum (-1)^s m^N_{i+l,j+k}(x^{\odot l_0} \odot \mathbf{1}_{\mathcal{E}} \odot x^{\odot l_1} \odot \ldots \odot \mathbf{1}_{\mathcal{E}} \odot x^{\odot l_i} \odot \mathbf{1}_N \odot \mathbf{1}_{\mathcal{A}} \ldots \odot \mathbf{1}_{\mathcal{A}}),$$

where $\mathbf{1}_{\mathcal{E}}$ denotes the identity in the space of matrices $\mathsf{Mat}^{\mathbf{Z}\mathcal{A}}$ and the sign exponent is

$$s = \sum_{1\leq t\leq i} t \times l_t.$$

Let us now detail the map underlying the A_∞-functor y''

$$\dot{y}'' : \mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}.$$

It sends an iterated extension M, given by a sequence M_i, $r \leq i \leq l$, and a matrix δ^M, to the $\mathbb{A}$-module which is the sum

$$\dot{y}''M = \sum_{r\leq i\leq l} \dot{y}M_i.$$

The multiplications $m^{\dot{y}''M}_j$, $j \geq 1$, defining its structure as an $\mathcal{A}$-polydule, are the morphisms $m^{N''}_{0,j-1}$, $j \geq 1$, which is the sum

$$\sum_{l\geq 0} m^N_{l,j-1}(x^{\odot l} \odot \mathbf{1}_{\dot{y}''M} \odot \mathbf{1}^{\odot j-1}_{\mathcal{A}}) = \sum_{l\geq 0} m^N_{l,j-1}\Big([y(\delta^M)]^{\odot l} \odot \mathbf{1}_{\dot{y}''M} \odot \mathbf{1}^{\odot j-1}_{\mathcal{A}}\Big).$$

Note that even though $\dot{y}''M$ and $\dot{f}M$ are isomorphic as $\mathbb{A}$ -modules, they differ as $\mathcal{A}$-polydules. The $\mathcal{A}$-polydule $\dot{y}''M$ should be considered as the torsion(twisting?) of fM by $y(\delta^M)$. Now, let us consider the morphisms y''_i, $i \geq 1$, of the A_∞-functor y''. They are defined (5.3.0.3) by the relation

$$(y''_i)_j = m^{N''}_{i,j-1}.$$

In other words, the morphism y''_i, $i \geq 1$, sends an element of

$$\mathsf{tw}\mathcal{A}(M_{i-1}, M_i) \otimes \ldots \otimes \mathsf{tw}\mathcal{A}(M_0, M_1)$$

to the morphism of $\mathcal{A}$-polydules $\varphi : (\dot{y}''M_0) \to (\dot{y}''M_i)$ given by the sequence of morphisms $\varphi_j : (\dot{y}''M_0) \odot \mathcal{A}^{\odot j-1} \to (\dot{y}''M_i)$ defined by (expressed as?)

$$\sum_{l\geq 0} \sum (-1)^s m^N_{i+l,j-1}\Big([y(\delta^{M_i})]^{\odot l_0} \odot \mathbf{1}_{\mathsf{tw}\mathcal{A}} \ldots \odot \mathbf{1}_{\mathsf{tw}\mathcal{A}} \odot [y(\delta^{M_0})]^{\odot l_i} \odot \mathbf{1}_{\dot{y}''M_0} \odot \mathbf{1}^{\odot j-1}_{\mathcal{A}}\Big),$$

where $\mathbf{1}_{\mathsf{tw}\mathcal{A}}$ denotes the identity in the space of matrices $\mathsf{Mat}^{\mathbf{Z}\mathcal{A}}$ and the sign exponent is

$$s = \sum_{1\leq t\leq i} t \times l_t.$$

Note that the strict unitality of $\mathcal{A}$ did not play a role in the proof of the factorization of the theorem 7.1.0.4. It plays an essential role in the next section.

7.4 The equivalence between the categories $\mathsf{tria}\,\mathcal{A}$ and $H^0\mathsf{tw}\mathcal{A}$

We recall (5.2.0.2) that the categories $H^0\mathcal{C}_\infty\mathcal{A}$ and $\mathcal{D}_\infty\mathcal{A}$ are equivalent. We show below that the A_∞-functor $y'' : \mathsf{tw}\mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$ induces a fully faithful functor

$$H^0\mathsf{tw}\mathcal{A} \to \mathcal{D}_\infty\mathcal{A}.$$

whose image is the category $\mathsf{tria}\,\mathcal{A}$.

The task is to show that the functor H^0y'' is fully faithful. Thus, we need to show that for all objects M, M' of $\mathsf{tw}\mathcal{A}$, we have

$$H^0\mathsf{Hom}_{\mathsf{tw}\mathcal{A}}(M, M') \xrightarrow{\sim} H^0\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(\dot{y}''M, \dot{y}''M').$$

An extension M, given by a sequence

$$(M_r, \ldots, M_i, \ldots, M_l), \quad r \leq i \leq l,$$

and a matrix δ^M, is clearly filtered in the category of twisted objects $\mathsf{tw}\mathcal{A}$ by

$$F_k = (M_{r+k}, \ldots, M_l), \quad 0 \leq k \leq l - r,$$

(The morphism $\delta^M : M \to M$ is compatible with this filtration). The graded objects of this filtration are degenerate twisted extensions, i.e., finite formal sums of $\mathbf{Z}\mathcal{A}$ considered as objects of $\mathsf{tw}\mathcal{A}$. Therefore, it suffices to show that there is an isomorphism

$$H^0\mathsf{Hom}_{\mathsf{tw}\mathcal{A}}(M, M') = H^0\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(\dot{y}''M, \dot{y}''M').$$

where M and M' are objects of $\mathbf{Z}\mathcal{A}$ considered as objects in $\mathsf{tw}\mathcal{A}$. We thus need to show the following lemma

Lemma 7.4.0.1. For any pair of objects A and A' in $\mathcal{A}$, the Yoneda A_∞-functor $y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}$ induces an isomorphism

$$H^*\mathsf{Hom}_{\mathcal{A}}(A, A') = H^*\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(A^\wedge, A'^\wedge).$$

Proof. The fully faithful functor (4.1.2.10)

$$\mathcal{D}_\infty\mathcal{A} \to \mathcal{D}_\infty\mathcal{A}^+$$

induces an isomorphism

$$H^*\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(A^\wedge, A'^\wedge) \xrightarrow{\sim} H^*\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}^+}(A^\wedge, A'^\wedge).$$

It suffices to show the isomorphism

$$H^*\mathcal{A}(A, A') \xrightarrow{\sim} H^*\mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}^+}(A^\wedge, A'^\wedge).$$

We have the equalities

$$\mathcal{A}(A, A') = A'^\wedge(A) \quad \text{and} \quad \mathsf{H\overset{\infty}{o}m}_{\mathcal{A}}(\mathcal{A}, A'^\wedge)(A) = \mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}^+}(A^\wedge, A'^\wedge).$$

We can then deduce the result from Lemma 4.1.1.6 and Remark 4.1.1.7 which show that

$$A'^\wedge \to \mathsf{H\overset{\infty}{o}m}_{\mathcal{A}}(\mathcal{A}, A'^\wedge)$$

is a quasi-isomorphism. □

7.5 Differential graded modules

In this section, the base category C is equal to $\mathsf{C}(\mathbb{A}, \mathbb{A})$.

Definition 7.5.0.1. Let $\mathcal{A}$ be an A_∞-algebra in C. A *differential graded module* $\mathcal{A}'$ of $\mathcal{A}$ is a differential graded algebra $\mathcal{A}'$ endowed with an A_∞-quasi-isomorphism

$$\mathcal{A} \to \mathcal{A}'.$$

Proposition 7.5.0.2. Every strictly unital A_∞-algebra $\mathcal{A}$ admits a unital differential graded module such that the A_∞-morphism

$$\mathcal{A} \to \mathcal{A}'$$

is strictly unital.

Note that in the case where $\mathcal{A}$ is an augmented A_∞-algebra, its enveloping algebra $U\mathcal{A}$ (2.3.4.3) is a unital differential graded module of $\mathcal{A}$ which is augmented.

Proof. We define $\mathcal{A}'$ as the $\mathbb{A}$-$\mathbb{A}$-bimodule

$$(A_0, A_1) \mapsto \mathsf{Hom}_{\mathcal{C}_\infty\mathcal{A}}(A_0{}^\wedge, A_1{}^\wedge).$$

The differential graded structure is the one induced by the composition and differential of the differential graded category $\mathcal{C}_\infty\mathcal{A}$. Thanks to Theorem (7.1.0.4), the A_∞-Yoneda functor gives us an A_∞-quasi-isomorphism between A_∞-algebras in $\mathsf{C}(\mathbb{A}, \mathbb{A})$

$$\mathcal{A} \to \mathcal{A}'$$

which is strictly unital. □

Corollary 7.5.0.3. Every homologically unital A_∞-algebra $\mathcal{A}$ admits a unital differential graded module such that the A_∞-morphism

$$f : \mathcal{A} \to \mathcal{A}'$$

is unital, i.e. $f \circ \eta = \eta$.

Proof. Let $\mathcal{A}$ be a homologically unital A_∞-algebra. We recall (3.2.1.2) that we can equip $H^*\mathcal{A}$ with a unital A_∞-algebra structure. As the A_∞-morphism

$$\mathcal{A} \to H^*\mathcal{A}$$

is unital and is an A_∞-quasi-isomorphism, we have the result. □

7.6 Stable categories

In this section, we show that every algebraic triangulated category generated by a set of objects is A_∞-pre-triangulated, i.e. it is equivalent to $H^0\mathsf{tw}\mathcal{A}$, for a certain A_∞-category $\mathcal{A}$.

Definition 7.6.0.1. A triangulated $\mathbb{K}$-category is *algebraic* if it is equivalent as a stable category to a Frobenius $\mathbb{K}$-category (see 2.2.3).

Definition 7.6.0.2. Let $\mathcal{T}$ be a triangulated category with infinite sums. An object $X \in \mathcal{T}$ is *compact* if the functor $\mathsf{Hom}_{\mathcal{T}}(X, -)$ commutes with infinite sums.

Definition 7.6.0.3. Let $\mathcal{T}$ be a triangulated category and $\mathbb{A}$ a subset of the set $\mathbb{T}$ of objects of $\mathcal{T}$. We denote by $\mathsf{tria}\,\mathbb{A}$ the *smallest strictly full triangulated subcategory* of $\mathcal{T}$ which contains the full subcategory formed by the objects of $\mathbb{A}$. It is stable under finite direct sums. The objects of $\mathbb{A}$ *generate* $\mathcal{T}$ as a triangulated category if $\mathcal{T} = \mathsf{tria}\,\mathbb{A}$. If $\mathcal{T}$ admits infinite sums, we denote by $\mathsf{Tria}\,\mathbb{A}$ the *smallest triangulated subcategory stable(closed?) under infinite sums* of $\mathcal{T}$ which contains the full subcategory consisting of objects of $\mathbb{A}$. The objects of $\mathbb{A}$ *generate* $\mathcal{T}$ as a triangulated category with infinite sums if $\mathcal{T} = \mathsf{Tria}\,\mathbb{A}$.

Theorem 7.6.0.4. Let $\mathcal{T}$ be an algebraic triangulated $\mathbb{K}$-category with infinite sums, generated, as a triangulated category with infinite sums, by a set $\mathbb{A}$ of compact objects. There exists an A_∞-category $\mathcal{A}$ that is strictly unital and minimal over $\mathbb{A}$, and a triangulated equivalence

$$\mathcal{D}_\infty\mathcal{A} \to \mathcal{T}, \quad A^\wedge \mapsto A.$$

Proof. By definition of algebaic triangulated categories, $\mathcal{T}$ is the stable category $\underline{\mathcal{E}}$ of a Frobenius category $\mathcal{E}$. We recall [Kel94a, 4.3] that there exists a unital differential graded category $\mathcal{A}'$ over $\mathbb{A}$ and a triangulated equivalence

$$\mathcal{D}\mathcal{A}' \to \underline{\mathcal{E}}, \quad A^\wedge \mapsto A.$$

Recall that $\mathcal{D}\mathcal{A}'$ is generated by the free A-modules $\mathcal{A}'(-, A)$, $A \in \mathbb{A}$. Let us choose a minimal model $\mathcal{A}$ of $\mathcal{A}'$ which is strictly unital (3.2.4.1). From Theorem (4.1.2.4), we deduce that the restriction along $\mathcal{A}' \to \mathcal{A}$ induces an equivalence of categories

$$\mathcal{D}_\infty\mathcal{A} \to \mathcal{D}\mathcal{A}'.$$

Since, for every $A \in \mathbb{A}$, the restricted $\mathcal{A}'$-polydue $A^\wedge = \mathcal{A}(-, A)$ is A_∞-quasi-isomorphic to $\mathcal{A}'(-, A)$, we have an equivalence

$$\mathcal{D}_\infty\mathcal{A} \to \underline{\mathcal{E}}, \quad A^\wedge \mapsto A.$$

□

Remark 7.6.0.5. By construction of the category $\mathcal{A}'$ in [Kel94a, 4.3], the $\mathbb{A}$-$\mathbb{A}$-bimodule underlying $\mathcal{A}$ is given by

$$(A, A') \mapsto \mathcal{A}(A, A') = \bigoplus_{n \in \mathbf{Z}} \mathsf{Hom}_{\mathcal{T}}(A, S^n A'), \quad A, A' \in \mathbb{A},$$

and $m_2^{\mathcal{A}}$ by the composition of $\mathcal{T}$.

Theorem 7.6.0.6. Let $\mathcal{T}$ be an algebraic triangulated $\mathbb{K}$-category which is generated by a set of objects $\mathbb{A}$. There exists an A_∞-category $\mathcal{A}$, strictly unital and minimal over $\mathbb{A}$, and a triangulated equivalence

$$\mathsf{tria}\, \mathcal{A} \to \mathcal{T}, \quad A^\wedge \mapsto A,$$

where $\mathsf{tria}\, \mathcal{A}$ is the subcategory of $\mathcal{D}_\infty \mathcal{A}$ generated by the free objects $A^\wedge$, $A \in \mathbb{A}$.

Proof. By definition of algebraic triangulated categories, $\mathcal{T}$ is the stable category $\underline{\mathcal{E}}$ of a Frobenius category $\mathcal{E}$. The construction of [Kel94a, 4.3] gives us a unital differential graded category $\mathcal{A}'$ over $\mathbb{A}$ such that we have a triangulated equivalence

$$\mathsf{tria}\, \mathcal{A}' \to \underline{\mathcal{E}}, \quad A^\wedge \mapsto A,$$

where $\mathsf{tria}\, \mathcal{A}'$ is the subcategory of $\mathcal{D}\mathcal{A}'$ generated by the free A-modules $\mathcal{A}'(-, A)$, $A \in \mathbb{A}$. Choose a minimal model $\mathcal{A}$ of $\mathcal{A}'$ which is strictly unital (3.2.4.1). The equivalence of categories

$$\mathcal{D}\mathcal{A}' \to \mathcal{D}_\infty \mathcal{A}.$$

induces an equivalence

$$\mathsf{tria}\, \mathcal{A}' \to \mathsf{tria}\, \mathcal{A}$$

because the $\mathcal{A}$-polydule $A^\wedge = \mathcal{A}(-, A)$, $A \in \mathbb{A}$ is A_∞-quasi-isomorphic to the restriction of $\mathcal{A}'(-, A)$. We deduce that we have a (triangulated) equivalence

$$\mathsf{tria}\, \mathcal{A} \to \underline{\mathcal{E}}, \quad A^\wedge \mapsto A.$$

□

Corollary 7.6.0.7. Let $\mathcal{T}$ be an algebraic triangulated $\mathbb{K}$-category, as in Theorem (7.6.0.6). There exists a strictly unital and minimal A_∞-category $\mathcal{A}$ over $\mathbb{A}$ and a triangulated equivalence

$$H^0(\mathsf{tw}\mathcal{A}) \to \mathcal{T}, \quad A \mapsto A.$$

Proof. Immediate by the theorems (7.1.0.4) and (7.6.0.6). □

Chapter 8

The A_∞-category of A_∞-functors

Introduction

The goal of this chapter is to construct the analog A_∞ of the 2-category cat of small categories. We construct a 2-category cat_∞ whose objects are the strictly unital A_∞-categories. The category of morphism spaces

$$\mathsf{cat}_\infty(\mathcal{A},\mathcal{B}),\quad \mathcal{A},\mathcal{B}\in \mathsf{Obj}\,\mathsf{cat}_\infty,$$

will be defined as the homology $H^0\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$ of an A_∞-category whose objects are the strictly unital A_∞-functors.

The category $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$ was constructed independently by K. Fukaya [Fuk01b], V. Lyubashenko [Lyu02] and M. Kontsevich and Y. Soibelman [KS02a], [KS02b]. The compositions of $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$ by V. Lyubashenko, although obtained by a different method, are the same as ours.

Chapter plan

Let $\mathcal{A}$ and $\mathcal{B}$ be two small A_∞-categories (not necessarily unital). In the section 8.1.1, we construct an A_∞-category $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$ whose objects are the A_∞-functors $\mathcal{A}\to\mathcal{B}$. The compositions of $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$ will be constructed by a process of torsion (see chapter 6). In the section 8.1.2, we show that $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$ is functorial in $\mathcal{A}$ and $\mathcal{B}$ and we define the category nat_∞ whose objects are the A_∞-categories. In section 8.1.3, we show that all the constructions of the two previous sections are compatible with strictly unital A_∞-structures (A_∞-categories, A_∞-functors...) and we define the 2-category cat_∞ as a non-full subcategory of nat_∞.

In the section (8.2), we build an A_∞-functor

$$z : \mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to \mathcal{C}_\infty(\mathcal{A},\mathcal{B}),\quad \mathcal{A},\mathcal{B}\in\mathsf{cat}_\infty,$$

where $\mathcal{C}_\infty(\mathcal{A},\mathcal{B})$ is the differential graded category of strictly unital $\mathcal{A}$-$\mathcal{B}$-bipolydules (8.2.1). This functor generalizes the A_∞-functor of Yoneda built in (7.1.0.1). We will show that it induces quasi-isomorphisms in the spaces of morphisms. In the section 8.2.2, we define the *weak equivalences* of strictly unital A_∞-functors (they are the A_∞-categorical analogue of the *homotopies* between A_∞-morphisms) and we will characterize them using their images by the A_∞-functor z.

8.1 The A$_\infty$-category of A$_\infty$-functors

8.1.1 The A$_\infty$-category $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets and $\mathcal{A}$ and $\mathcal{B}$ be two A$_\infty$-categories over $\mathbb{A}$ and $\mathbb{B}$. We construct in this section the A$_\infty$-category $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$ of not necessarily strictly unital A$_\infty$-functors. The letter N replaces the letter F in Func_∞ and refers to the N of "Non unital".

Let f_1 and f_2 be two A$_\infty$-functors $\mathcal{A}\to\mathcal{B}$. We recall that ${}_{\dot f_2}\mathcal{B}_{\dot f_1}$ is the $\mathbb{A}$-$\mathbb{A}$-bimodule

$$(A',A)\mapsto \mathcal{B}(\dot f_1 A',\dot f_2 A).$$

Definition 8.1.1.1. We set

$$\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1,f_2)=\mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(T^cS\mathcal{A},{}_{\dot f_2}\mathcal{B}_{\dot f_1}\big).$$

We thus obtain a graded object in the base category $\mathsf{Vect}\mathbb{K}$.

Remark 8.1.1.2. Let H be an element of degree r of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1,f_2)$. For any integer $i\geq 0$, we denote by incl the inclusion of $(S\mathcal{A})^{\odot i}$ in $T^cS\mathcal{A}$. Let H_i, $i\geq 0$ be the composition

$$(S\mathcal{A})^{\odot i}\xrightarrow{\mathsf{incl}} T^cS\mathcal{A}\xrightarrow{h} {}_{\dot f_2}\mathcal{B}_{\dot f_1}.$$

We define the morphisms

$$h_i:\mathcal{A}^{\odot i}\to {}_{\dot f_2}\mathcal{B}_{\dot f_1},\quad i\geq 0,$$

by the relations

$$H_i\circ(\omega^{\odot i})^{-1}=(-1)^r h_i,\quad i\geq 0.$$

The maps $H_i\mapsto h_i$, $i\geq 0$, are clearly bijections. The morphism H is therefore determined by graded morphisms

$$h_i:\mathcal{A}(A_{i-1},A_i)\otimes\ldots\otimes\mathcal{A}(A_0,A_1)\to\mathcal{B}(\dot f_1 A_0,\dot f_2 A_i),\quad i\geq 0.$$

of degree $r-i$, for any sequence $(A_0,\ldots,A_i)$ of objects of $\mathcal{A}$. In particular, if $i=0$, we have a morphism

$$h_0:e_{\mathbb{A}}\to {}_{\dot f_2}\mathcal{B}_{\dot f_1},\quad \mathbf{I}_A\mapsto h_0(\mathbf{I}_A).$$

We will often denote $h_A\in\mathsf{Hom}_{\mathcal{B}}(\dot f_1 A,\dot f_2 A)$ the element $h_0(\mathbf{I}_A)$.

Remark 8.1.1.3. Let $f:\mathcal{A}\to\mathcal{B}$ be an A$_\infty$-functor. Let $h_i=f_i$ if $i\geq 1$ and $h_0=0$. This gives us an element H of degree $+1$ of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f,f)$. We then have a commutative diagram

$$\begin{tikzcd}
(S\mathcal{A})^{\odot i} \arrow[r, hook] \arrow[dr, "H_i"'] & B^+\mathcal{A} \arrow[r, "F"] \arrow[d, "H"] & B^+{}_{\dot f}\mathcal{B}_{\dot f} \arrow[d, "p_1"] \\
 & {}_{\dot f}\mathcal{B}_{\dot f} & S{}_{\dot f}\mathcal{B}_{\dot f} \arrow[l, "\omega"]
\end{tikzcd}$$

from which we deduce the equalities $H_i=\omega\circ F_i$, where F is the co-augmented bar construction of f.

Naive compositions of morphisms of A_∞-functors

We construct in this paragraph an A_∞-category $\mathcal{F}(\mathcal{A},\mathcal{B}) = \mathcal{F}$ whose objects are the A_∞-functors $\mathcal{A} \to \mathcal{B}$ and whose graded morphism spaces are

$$\mathsf{Hom}_{\mathcal{F}}(f_1, f_2) = \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(T^c S\mathcal{A}, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\big).$$

We show that $\mathcal{F}$ is endowed with a topology for which it is a topological A_∞-category. We then construct a topological twisting element of $\mathcal{F}$ (see 6.2).

Instead of constructing the compositions $m_i^{\mathcal{F}}$, $i \geq 1$, we will construct morphisms (see the bijections $m_i \leftrightarrow b_i$ in the section 1.2.2)

$$b_i^{\mathcal{F}} : S\mathcal{F}^{\odot i} \to S\mathcal{F}, \quad i \geq 1,$$

then we check that it defines an A_∞-category. Note that we have an isomorphism

$$S\mathcal{F}(f_1, f_2) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(B^+\mathcal{A}, S_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\big).$$

The morphism

$$b_1^{\mathcal{F}} : \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(B^+\mathcal{A}, S_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\big) \to \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(B^+\mathcal{A}, S_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\big)$$

is the differential of the graded morphism spaces between complexes: it is defined by

$$\varphi \mapsto b_1^{\mathcal{B}} \circ \varphi - (-1)^{|\varphi|} \varphi \circ b^{B^+\mathcal{A}},$$

where φ is of degree $|\varphi|$. Let $i \geq 2$ and $(f_0, \ldots, f_i)$ be a sequence of A_∞-functors $\mathcal{A} \to \mathcal{B}$. The morphism $b_i^{\mathcal{F}}$ sends an element

$$g_i \odot \ldots \odot g_1 \in \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(B^+\mathcal{A}, S_{\dot{f}_i}\mathcal{B}_{\dot{f}_{i-1}}\big) \odot \ldots \odot \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\big(B^+\mathcal{A}, S_{\dot{f}_1}\mathcal{B}_{\dot{f}_0}\big)$$

to the composition

$$B^+\mathcal{A} \xrightarrow{\Delta^{(i)}} (B^+\mathcal{A})^{\odot i} \xrightarrow{g_i \odot \ldots \odot g_1} S_{\dot{f}_i}\mathcal{B}_{\dot{f}_{i-1}} \odot \ldots \odot S_{\dot{f}_1}\mathcal{B}_{\dot{f}_0} \xrightarrow{b_i^{\mathcal{B}}} S_{\dot{f}_i}\mathcal{B}_{\dot{f}_0}.$$

Lemma 8.1.1.4. The morphisms $m_i^{\mathcal{F}}$, $i \geq 1$, define an A_∞-category structure on $\mathcal{F}$.

Proof. We clearly have $b_1^{\mathcal{F}} \circ b_1^{\mathcal{F}} = 0$. Let $n \geq 2$ and let g_i, $1 \leq i \leq n$ be elements of $S\mathcal{F}$ of degree $|g_i|$. The terms of the sum

$$\Big[\sum_{j+k+l=n} b_i^{\mathcal{F}}(\mathbf{I}^{\odot j} \odot b_k^{\mathcal{F}} \odot \mathbf{I}^{\odot l})\Big](g_n \odot \ldots \odot g_1)$$

are of three types: those where $i = n$ and $k = 1$, those where $i = 1$ and $k = n$ and those where $i, j \neq 1$.

- When $i = n$ and $k = 1$ we find

$$
\begin{aligned}
& \big[b_n^{\mathcal{F}}(\mathbf{I}^{\odot j} \odot b_1^{\mathcal{F}} \odot \mathbf{I}^{\odot l})\big](g_n \odot \ldots \odot g_1) \\
= \; & (-1)^{\sum_{r<l+1} |g_r|} b_n^{\mathcal{F}}(g_n \odot \ldots \odot b_1^{\mathcal{F}}(g_{l+1}) \odot \ldots \odot g_1) \\
= \; & (-1)^{\sum_{r<l+1} |g_r|} b_n^{\mathcal{F}}(g_n \odot \ldots \odot b_1^{\mathcal{B}} g_{l+1} \odot \ldots \odot g_1) \\
& -(-1)^{\sum_{r\leq l+1} |g_r|} b_n^{\mathcal{F}}(g_n \odot \ldots \odot g_{l+1} b^{B^+\mathcal{A}} \odot \ldots \odot g_1) \\
= \; & b_n^{\mathcal{B}}(\mathbf{I}^{\odot j} \odot b_1^{\mathcal{B}} \odot \mathbf{I}^{\odot l})(g_n \odot \ldots \odot g_1)\Delta^{(n)} \\
& -(-1)^{\sum_r |g_r|} b_n^{\mathcal{B}}(g_n \odot \ldots \odot g_{l+1} \odot \ldots \odot g_1)(\mathbf{I}^{\odot j} \odot b^{B^+\mathcal{A}} \odot \mathbf{I}^{\odot l})\Delta^{(n)} \\
= \; & b_n^{\mathcal{B}}(\mathbf{I}^{\odot j} \odot b_1^{\mathcal{B}} \odot \mathbf{I}^{\odot l})(g_n \odot \ldots \odot g_1)\Delta^{(n)} \\
& -(-1)^{\sum_r |g_r|} b_n^{\mathcal{B}}(g_n \odot \ldots \odot g_{l+1} \odot \ldots \odot g_1)\Delta^{(n)} b^{B^+\mathcal{A}} \\
= \; & b_n^{\mathcal{B}}(\mathbf{I}^{\odot j} \odot b_1^{\mathcal{B}} \odot \mathbf{I}^{\odot l})(g_n \odot \ldots \odot g_1)\Delta^{(n)} \\
& -(-1)^{\sum_r |g_r|} b_n^{\mathcal{F}}(g_n \odot \ldots \odot g_{l+1} \odot \ldots \odot g_1) b^{B^+\mathcal{A}}
\end{aligned}
$$

- When $i = 1$ and $k = n$ we find

$$
\begin{aligned}
& b_1^{\mathcal{F}} \cdot b_n^{\mathcal{F}}(g_n \odot \ldots \odot g_1) \\
= \; & b_1^{\mathcal{B}}(b_n^{\mathcal{F}}(g_n \odot \ldots \odot g_1)) \\
& -(-1)^{1+\sum_r |g_r|} b_n^{\mathcal{F}}(g_n \odot \ldots g_{l+1} \odot \ldots \odot g_1) b^{B^+\mathcal{A}} \\
= \; & b_1^{\mathcal{B}}(b_n^{\mathcal{B}}(g_n \odot \ldots \odot g_1)\Delta^{(n)}) \\
& -(-1)^{1+\sum_r |g_r|} b_n^{\mathcal{F}}(g_n \odot \ldots g_{l+1} \odot \ldots \odot g_1) b^{B^+\mathcal{A}}
\end{aligned}
$$

- When $i \neq 1$ and $k \neq n$ we find

$$
\begin{aligned}
& \big[b_i^{\mathcal{F}}(\mathbf{I}^{\odot j} \odot b_k^{\mathcal{F}} \odot \mathbf{I}^{\odot l})\big](g_n \odot \ldots \odot g_1) \\
= \; & (-1)^{\sum_{r<l+1} |g_r|} b_i^{\mathcal{F}}(g_n \odot \ldots b_j^{\mathcal{F}}(g_{l+j+1} \odot \ldots \odot g_{l+1}) \odot \ldots \odot g_1) \\
= \; & (-1)^{\sum_{r<l+1} |g_r|} b_i^{\mathcal{F}}(g_n \odot \ldots (b_j^{\mathcal{B}}(g_{l+j+1} \odot \ldots \odot g_{l+1})\Delta^{(k)}) \odot \ldots \odot g_1) \\
= \; & (-1)^{\sum_{r<l+1} |g_r|} b_i^{\mathcal{B}}(g_n \odot \ldots (b_j^{\mathcal{B}}(g_{l+j+1} \odot \ldots \odot g_{l+1})\Delta^{(k)}) \odot \ldots \odot g_1)\Delta^{(i)} \\
= \; & b_i^{\mathcal{B}}(\mathbf{I}^{\odot j} \odot b_j^{\mathcal{B}} \odot \mathbf{I}^{\odot l})(g_n \odot \ldots \odot g_1)(\mathbf{I}^{\odot j} \odot \Delta^{(k)} \odot \mathbf{I}^{\odot l})\Delta^{(i)} \\
= \; & b_i^{\mathcal{B}}(\mathbf{I}^{\odot j} \odot b_j^{\mathcal{B}} \odot \mathbf{I}^{\odot l})(g_n \odot \ldots \odot g_1)\Delta^{(n)}
\end{aligned}
$$

The last lines of the first two cases compensate each other thanks to the signs and the sum of what remains is zero because $\mathcal{B}$ is an A$_\infty$-category. □

Remark 8.1.1.5. The A$_\infty$-category $\mathcal{F}(\mathcal{A}, \mathcal{B})$ thus constructed is clearly functorial in $\mathcal{A}$ and $\mathcal{B}$. If $f : \mathcal{A} \to \mathcal{A}'$ is an A$_\infty$-functor, the induced A$_\infty$-functor $\mathcal{F}(\mathcal{A}', \mathcal{B}) \to \mathcal{F}(\mathcal{A},\ cb)$ is strict. It sends

$H \in \mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$ to its composition with Bf. If $f : \mathcal{B} \to \mathcal{B}'$ is an A$_\infty$-functor, the induced A$_\infty$-functor $\mathcal{F}(\mathcal{A}', \mathcal{B}) \to \mathcal{F}(\mathcal{A},\ cb)$ is no longer strict. Note the g. Let G be its bar construction. The morphism G_1 sends $H \in \mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$ to its composition with F_1. The formulas defining the G_i, $i \geq 2$, are obtained from the formulas defining the $b_i^{\mathcal{F}}$, $i \geq 2$, by replacing the $b_i^{\mathcal{B}}$ by F_i. Functoriality issues will be studied in more detail in section 8.1.2.

Concrete description

Let's look at what are the compositions of morphisms of A$_\infty$-categories from the point of view of the remark 8.1.1.2.

Let H be an element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$ of degree $|H|$. The morphism $m_1^{\mathcal{F}}(H)$ is determined by morphisms

$$h_i' : \mathcal{A}^{\odot i} \to {}_{f_2}\mathcal{B}_{f_1}, \quad i \geq 0.$$

We check that h_i' is equal to the sum

$$m_1^{\mathcal{B}} \circ h_i - (-1)^{|H|} \sum (-1)^{l+kj} h_{j+1+l}(\mathbf{1}^{\odot j} \odot m_k^{\mathcal{A}} \odot \mathbf{1}^{\odot l}).$$

Let $i \geq 2$. Let $f_0, \ldots, f_i$ be A$_\infty$-functors $\mathcal{A} \to \mathcal{B}$. For all $1 \leq t \leq i$, let H_t be an element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_{t-1}, f_t)$ of degree $|H_t|$. Let $|H|$ be the sum of the degrees $|H_t|$. Let H' be the element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_0, f_i)$ equal to $m_i^{\mathcal{F}}(H_n \odot \ldots \odot H_1)$. Then H' is given by graded morphisms

$$h_n' : \mathcal{A}(A_{n-1}, A_n) \otimes \ldots \otimes \mathcal{A}(A_0, A_1) \to \mathcal{B}(\dot{f}_0 A_0, \dot{f}_i A_n), \quad n \geq 0.$$

of degree $|H| - n$, for any sequence $(A_0, \ldots, A_n)$ of objects of $\mathcal{A}$. Let $x_k \in \mathcal{A}(A_{k-1}, A_k)$, $1 \leq k \leq n$. We denote by incl the inclusion of $(S\mathcal{A})^{\odot i}$ in $B^+\mathcal{A}$. The element $h_n'(x_n \odot \ldots \odot x_1)$ is equal to

$$-\omega \circ b_i^{\mathcal{B}} \circ \left[(\omega^{\odot i})^{-1}(H_i \odot \ldots \odot H_1)\right] \circ \Delta^{(i)} \circ \mathsf{incl} \circ (\omega^{\odot n})^{-1}(x_n \odot \ldots \odot x_1)$$

Let's take a simple example.

Example 8.1.1.6. Suppose $i = 3$ and $n = 2$. The composition $\Delta^{(3)} \circ \mathsf{incl} \circ (\omega^{\odot 2})^{-1}(x_2 \odot x_1)$ is equal to the sum in $B^+\mathcal{A}3$

$$\begin{aligned}\big[&\mathbf{I}_{A_2} \odot \mathbf{I}_{A_2} \odot (\omega^{\odot 2})^{-1} - \mathbf{I}_{A_2} \odot (\omega)^{-1} \odot (\omega)^{-1} - \\ &(\omega)^{-1} \odot \mathbf{I}_{A_1} \odot (\omega)^{-1} + \mathbf{I}_{A_2} \odot (\omega^{\odot 2})^{-1} \odot \mathbf{I}_{A_0} \\ &-(\omega)^{-1} \odot (\omega)^{-1} \odot \mathbf{I}_{A_0} + (\omega^{\odot 2})^{-1} \odot \mathbf{I}_{A_0} \odot \mathbf{I}_{A_0}\big](x_2 \odot x_1).\end{aligned}$$

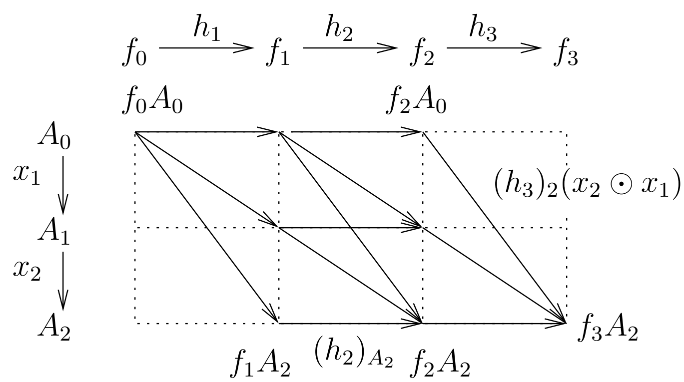
We therefore find that $m_3^{\mathcal{F}}(h_3 \odot h_2 \odot h_1)(x_2 \odot x_1)$ is equal to the sum of the elements

$$\begin{aligned} m_3^{\mathcal{B}}\Big(&\pm (h_3)_{A_2} \odot (h_2)_{A_2} \odot (h_1)_2 \pm (h_3)_{A_2} \odot (h_2)_1 \odot (h_1)_1 \\ &\pm (h_3)_1 \odot (h_2)_{A_1} \odot (h_1)_1 \pm (h_3)_{A_2} \odot (h_2)_2 \odot (h_1)_{A_0} \\ &\pm (h_3)_1 \odot (h_2)_1 \odot (h_1)_{A_0} \pm (h_3)_2 \odot (h_2)_{A_0} \odot (h_1)_{A_0}\Big)(x_2 \odot x_1).\end{aligned}$$

The morphism

$$h_2'(x_2 \odot x_1) : f_0 A_0 \to f_3 A_2$$

is therefore the sum of the compositions (up to signs) of the sequences of morphisms represented by a path of arrows leading from $f_0 A_0$ to $f_3 A_2$ in the diagram below



.

Note that there is no vertical arrow (which would correspond to a $(f_j)_1(x_i)$ or a $(f_j)_2(x_2 \otimes x_1)$) in these paths of arrows.

Generally speaking, we find that the element H' of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_0, f_n)$ is given by

$$h'_n = \sum_{j_1+..+j_l=n} (-1)^s m_l^{\mathcal{B}}\big((h_i)_{j_1} \odot \ldots \odot (h_1)_{j_l}\big), \quad n \geq 0,$$

where the integers j_α are ≥ 0, and where the sign is given by the equality

$$(-1)^s\big((H_i)_{j_1} \odot \ldots \odot (H_1)_{j_l}\big) \circ (\omega^{\odot n}) = \big((h_i)_{j_1} \odot \ldots \odot (h_1)_{j_l}\big).$$

Remark 8.1.1.7. Let H be the element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f, f)$ constructed in remark (8.1.1.3). If $f_t = f$, $0 \leq t \leq i$, and $H_t = H$, $1 \leq t \leq i$, the sign $(-1)^s$ above is the same as the sign $(-1)^s$ of the equation $(**_n)$, $n \geq 1$, in the definition of A$_\infty$-functors (5.1.2.5).

Topology on $\mathcal{F}$

We equip the space

$$\mathsf{Hom}_{\mathcal{F}}(f_1, f_2) = \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}(B^+\mathcal{A}, {}_{f_2}\mathcal{B}_{f_1})$$

of the topology defined by the decreasing filtration F_i, $i \geq 0$, where

$$F_i = \mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}\Big(\bigoplus_{j\geq i}(S\mathcal{A})^{\odot j}, {}_{f_2}\mathcal{B}_{f_1}\Big).$$

This topology is separated. The above description shows that the compositions of $\mathcal{F}$ are contracting continuous morphisms (see 6.2.1). The A$_\infty$-category $\mathcal{F}$ is therefore topological (6.2.1.1).

Twisting element of $\mathcal{F}$

Let $\mathbb{F}$ denote the set of A$_\infty$-functors $\mathcal{A} \to \mathcal{B}$. The twisting element

$$x : e_{\mathbb{F}} \to \mathcal{F}$$

sends the generator $\mathbf{I}_f$ of $e_{\mathbb{F}}(f, f)$ on the element H of degree $+1$ of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f, f)$ constructed from f (see 8.1.1.3).

Let us now check that x is a topological twisting element. As the morphism h_0 is null, the image of x is in the neighborhood $\mathcal{F}_1$. The restriction of the sum

$$\sum_{i\geq 1} m_i^{\mathcal{F}}(H^{\odot i})(\mathbf{I}_f) : B^+\mathcal{A} \to {}_{\dot{f}}\mathcal{B}_{\dot{f}}$$

at $(S\mathcal{A})^{\odot n}$ is the sum

$$-\sum(-1)^{jk+l} h_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) + \sum_{j_1+..+j_l=n} (-1)^s m_l^{\mathcal{B}}\big(h_{j_1} \odot \ldots \odot h_{j_l}\big)$$

Recall that $h_i = f_i$, $i \geq 1$. The Maurer-Cartan equation applied to $\mathbf{I}_f$ is therefore equivalent to the set of equations $(**_n)$, $n \geq 1$, of the definition of an A$_\infty$-functor (5.1.2.5).

The A$_\infty$-category $\mathsf{Nunc}_\infty(\mathcal{A}, \mathcal{B})$

Definition 8.1.1.8 (See also [Fuk01b], [Lyu02] and [KS02a], [KS02b]).
The A$_\infty$-*category* $\mathsf{Nunc}_\infty(\mathcal{A}, \mathcal{B})$ is the twisted category $\mathcal{F}_x$ (see 6.2.4.3 for the twist).

Note that the compositions $m_i^{\mathsf{Nunc}_\infty}$, $i \geq 1$, of [Fuk01b], [Lyu02] are the same but obtained in different ways.

Concrete description

Let us now give a description of the morphism

$$m_1^{\mathsf{Nunc}_\infty} : \mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2) \to \mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2).$$

Let H be an element of degree $|H|$ of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$. The morphism $H' = m_1^{\mathsf{Nunc}_\infty}(H)$ is determined by morphisms

$$h'_i : \mathcal{A}^{\odot i} \to {}_{f_2}\mathcal{B}_{f_1}, \quad i \geq 0.$$

We check that h'_i is equal to the sum

$$\begin{aligned}\textstyle\sum_{j_1+..+j_l=n} &(-1)^s m_l^{\mathcal{B}}\big((f_2)_{j_1} \odot \ldots \odot (f_2)_{j_t} \odot h_{j_{t+1}} \odot (f_1)_{j_{t+1}} \ldots \odot (f_1)_{j_l}\big)\\ &-(-1)^{|h|+l+kj} h_{j+1+l}(\mathbf{1}^{\odot j} \odot m_k^{\mathcal{A}} \odot \mathbf{1}^{\odot l}),\end{aligned}$$

where the exponent of the sign s is the sum of the sign appearing in the torsion (6.1.2) and the sign given by the equality

$$\begin{aligned}(-1)^*\big((\omega F_2)_{j_1} \odot \ldots \odot (\omega F_2)_{j_t} \odot H_{j_{t+1}} \odot (\omega F_1)_{j_{t+1}} \ldots \odot (\omega F_1)_{j_l}\big) \circ (\omega^{\odot n})\\ = \big((f_2)_{j_1} \odot \ldots \odot (f_2)_{j_t} \odot h_{j_{t+1}} \odot (f_1)_{j_{t+1}} \ldots \odot (f_1)_{j_l}\big).\end{aligned}$$

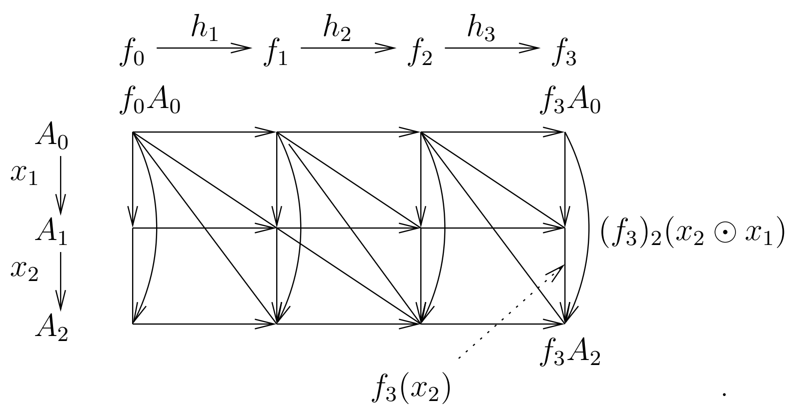
The description of the upper compositions $m_i^{\mathsf{Nunc}_\infty}$, $i \geq 2$, is done in a similar way. Let's go back to the 8.1.1.6 example and set

$$H'' = m_3^{\mathsf{Nunc}_\infty}(h_3 \odot h_2 \odot h_1) \in \mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_0, f_3).$$

The morphism

$$h''_2(x_2 \odot x_1) : f_0A_0 \to f_3A_2$$

is the sum of the compositions (up to signs) of the sequences of morphisms represented by a path of arrows leading from f_0A_0 to f_3A_2 in the diagram below



.

Graphically, the torsion consists of allowing vertical arrows in the paths.

Remark 8.1.1.9. If $\mathcal{B}$ is a differential graded category, the category $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$ is also a differential graded category because the compositions $m_i^{\mathsf{Nunc}_\infty}$, $i \geq 3$ are null.

8.1.2 Functoriality of $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$

Functoriality in $\mathcal{A}$

Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ be small A$_\infty$-categories. Let $g \in \mathcal{A}' \to \mathcal{A}$, $f_1, f_2 : \mathcal{A} \to \mathcal{B}$ be A$_\infty$-functors. Let H be an element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$. We define the element

$$H \star g \in \mathsf{Hom}_{\mathsf{Nunc}_\infty}((f_1 \circ g), (f_2 \circ g))$$

as the composition

$$B^+\mathcal{A}' \xrightarrow{G} B^+{}_{\dot{g}}\mathcal{A}_{\dot{g}} \to {}_{\dot{f}_2\dot{g}}\mathcal{B}_{\dot{f}_1\dot{g}}$$

where the second arrow is induced by H. As G is a morphism of differential graded coalgebras, the morphism of $\mathbb{F}$-$\mathbb{F}$-bimodules

$$? \star g : \mathsf{Nunc}_\infty(f_1, f_2) \to \mathsf{Nunc}_\infty((f_1 \circ g), (f_2 \circ g))$$

is a strict A$_\infty$-functor.

Functoriality in $\mathcal{B}$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}'$ be small A$_\infty$-categories. Let $g \in \mathcal{B} \to \mathcal{B}'$, $f_1, f_2 : \mathcal{A} \to \mathcal{B}$ be A$_\infty$-functors. Let H be an element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$. We will construct an element

$$g \star H \in \mathsf{Hom}_{\mathsf{Nunc}_\infty}((g \circ f_1), (g \circ f_2)).$$

This will provide us with a strict A$_\infty$-functor

$$g \star ? : \mathsf{Nunc}_\infty(f_1, f_2) \to \mathsf{Nunc}_\infty((g \circ f_1), (g \circ f_2))$$

Let's start by introducing a few concepts.

Let M be a differential graded $\mathbb{A}$-$\mathbb{A}$-bimodule. Let C, C_1 and C_2 cocomplete coalgebras in the category of differential graded coalgebras of the base category $\mathsf{C}(\mathbb{A},\mathbb{A})$. We endow the $\mathbb{A}$-$\mathbb{A}$-bimodule $C_2 \odot M \odot C_1$ with the structure of (cocomplete) C_2-C_1-bicomodule induced by the comultiplications of C_2 and C_1. Let

$$F_1 : C \to C_1 \quad \text{and} \quad F_2 : C \to C_2$$

be morphisms of coalgebras.

Definition 8.1.2.1. A *(F_1,F_2)-coderivation* is a morphism of $\mathbb{A}$-$\mathbb{A}$-bimodules

$$K : C \to C_2 \odot M \odot C_1$$

such that

$$(\Delta^{C_2} \odot \mathbf{1} \odot \mathbf{1}) \circ K = (F_2 \odot K) \circ \Delta^C \quad \text{and} \quad (\mathbf{1} \odot \mathbf{1} \odot \Delta^{C_1}) \circ K = (K \odot F_1) \circ \Delta^C.$$

Lemma 8.1.2.2. Let p_1 be the projection $C_2 \odot M \odot C_1$ onto M. The map $K \circ p_1 \circ K$ is a bijection of the set of (F_1,F_2)-coderivations to the morphisms of $\mathbb{A}$-$\mathbb{A}$-bimodules $C \to M$. $\square$

Let C_1, C_2 and C_3 be cocomplete coalgebras in the category of differential graded coalgebras of the base category $\mathsf{C}(\mathbb{A},\mathbb{A})$. The *cotensorial product* of a C_1-C_2-bicomodule M with a C_2-C_3-bicomodule N is the kernel

$$M \boxdot N = \mathsf{ker}\left(M \odot N \xrightarrow{\Delta\odot\mathbf{1}-\mathbf{1}\odot\Delta} M \odot C_2 \odot N\right).$$

Let's resume the construction of $H \star g$. We recall that the $\mathbb{A}$-$\mathbb{A}$-bimodules ${}_{\dot f_1}\mathcal{B}_{\dot f_1}$ and ${}_{\dot f_2}\mathcal{B}_{\ dotf_2}$ are A_∞-categories over $\mathbb{A}$. Let

$$F_1 : B^+\mathcal{A} \to B^+{}_{\dot f_1}\mathcal{B}_{\dot f_1} \quad \text{and} \quad F_2 : B^+\mathcal{A} \to B^+{}_{\dot f_2}\mathcal{B}_{\dot f_2}$$

the co-augmented bar construction of f_1 and f_2. The morphism

$$H : B^+\mathcal{A} \to {}_{\dot f_2}\mathcal{B}_{\dot f_1}$$

lifts to a (F_1,F_2)-coderivation of comodules

$$K : B^+\mathcal{A} \to B^+{}_{\dot f_2}\mathcal{B}_{\dot f_2} \odot {}_{\dot f_2}\mathcal{B}_{\dot f_1} \odot B^+{}_{\dot f_1}\mathcal{B}_{\dot f_1}.$$

The A_∞-functor $g : \mathcal{B} \to \mathcal{B}'$ induces a morphism G of degree 0

$$B^+{}_{\dot f_2}\mathcal{B}_{\dot f_2} \odot {}_{\dot f_2}\mathcal{B}_{\dot f_1} \odot B^+{}_{\dot f_1}\mathcal{B}_{\dot f_1} \to B^+{}_{\dot g\dot f_2}\mathcal{B}_{\dot g\dot f_2} \odot {}_{\dot g\dot f_2}\mathcal{B}_{\dot g\dot f_1} \odot B^+{}_{\dot g\dot f_1}\mathcal{B}_{\dot g\dot f_1}.$$

We verify that the composition $G \circ K$ defines a (GF_1, GF_2)-coderivation

$$B^+\mathcal{A} \to B^+{}_{\dot g\dot f_2}\mathcal{B}_{\dot g\dot f_2} \odot {}_{\dot g\dot f_2}\mathcal{B}_{\dot g\dot f_1} \odot B^+{}_{\dot g\dot f_1}\mathcal{B}_{\dot g\dot f_1}$$

and we define the element $g \star H$ by the composition

$$p_1 \circ (G \circ K) : B^+\mathcal{A} \to {}_{\dot g\dot f_2}\mathcal{B}_{\dot g\dot f_1}.$$

Equip $B^+{}_{\dot f_2}\mathcal{B}_{\dot f_2} \odot {}_{\dot f_2}\mathcal{B}_{\dot f_1} \odot B^+{}_{\dot f_1}\mathcal{B}_{\dot f_1}$ the differential induced by the $b_i^{\mathcal{B}}$, $i \geq 1$, and let $D(f_2, f_1)$ be the differential graded bicomodule obtained in this way. We can consider $D(f_1, f_2)$ as the bar construction of the ${}_{\dot f_2}\mathcal{B}_{\dot f_2}$-${}_{\dot f_1}\mathcal{B}_{\dot f_1}$-bipolydule ${}_{\dot f_2}\mathcal{B}_{\dot f_1}$.

Remark 8.1.2.3. Let H be an element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$ and K the associated coderivation. The element $m_1^{\mathsf{Nunc}_\infty}(H)$ corresponds to the coderivation $\delta(K)$ in the differential graded space of graded morphisms

$$\Big(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}(\mathbb{A},\mathbb{A})}(B^+\mathcal{A}, D(f_2, f_1)), \delta\Big).$$

Let $i \geq 2$. Let $f_0, \dots, f_i$ be A$_\infty$-functors $\mathcal{A} \to \mathcal{B}$. For all $1 \leq t \leq i$, let H_t be an element of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_{t-1}, f_t)$ of degree $|H_t|$. Let C_t be the differential graded coalgebra $B^+{}_{\dot{f}_t}\mathcal{B}_{\dot{f}_t}$. The C_i-C_0-bicomodule

$$D(f_i, f_{i-1}) \boxdot \cdots \boxdot D(f_1, f_0)$$

is isomorphic as a graded object to

$$C_i \odot {}_{\dot{f}_i}\mathcal{B}_{\dot{f}_{i-1}} \odot C_{i-1} \odot {}_{\dot{f}_{i-1}}\mathcal{B}_{\dot{f}_{i-2}} \odot C_{i-2} \odot \ldots \odot C_1 \odot {}_{\dot{f}_1}\mathcal{B}_{\dot{f}_0} \odot C_1.$$

We equip it with the differential induced by the $b_i^{\mathcal{B}}$, $i \geq 1$. The element

$$m_i(H_i \odot \ldots \odot H_1) : B^+\mathcal{A} \to {}_{\dot{f}_i}\mathcal{B}_{\dot{f}_0}$$

corresponds to the F_i-F_1-coderivation

$$K : B^+\mathcal{A} \to D(f_i, f_0)$$

which is the lifting

$$B^+\mathcal{A} \xrightarrow{\Delta^{(i)}} (B^+\mathcal{A})^{\odot i} \xrightarrow{K_i \boxdot \ldots \boxdot K_1} D(f_i, f_{i-1}) \boxdot \ldots \boxdot D(f_1, f_0) \xrightarrow{q} {}_{\dot{f}_i}\mathcal{B}_{\dot{f}_0},$$

where q is induced by the $b_i^{\mathcal{B}}$, $i \geq 1$.

The A$_\infty$-function g induces morphisms

$$D(f_i, f_{i-1}) \boxdot \ldots \boxdot D(f_1, f_0) \to D(gf_i, gf_{i-1}) \boxdot \ldots \boxdot D(gf_1, gf_0)$$

and a lift to $D(gf_i, gf_0)$ of

$$D(f_i, f_{i-1}) \boxdot \ldots \boxdot D(f_1, f_0) \longrightarrow {}_{\dot{g}\dot{f}_i}\mathcal{B}_{\dot{g}\dot{f}_0}$$

which are compatible with differentials. We deduce that the morphism of $\mathbb{F}$-$\mathbb{F}$-bimodules

$$g \star ? : \mathsf{Nunc}_\infty(f_1, f_2) \to \mathsf{Nunc}_\infty((g \circ f_1), (g \circ f_2))$$

defines a strict A$_\infty$-functor.

The category nat_∞

Let nat_∞ be the *category* whose objects are the small A$_\infty$-categories (not necessarily strictly unital), whose morphism spaces are the categories (without units in general)

$$\mathsf{nat}_\infty(\mathcal{A}, \mathcal{B}) = H^0\mathsf{Nunc}_\infty(\mathcal{A}, \mathcal{B}).$$

It follows from the functoriality of $\mathsf{Nunc}_\infty(\mathcal{A}, \mathcal{B})$ that nat_∞ is a unitless "2-category for 2-morphisms". The letter n replaces the letter c of cat_∞ and expresses the fact that the objects of nat_∞ are the A$_\infty$-"cat" egories "n"on (necessarily) strictly unital.

Remark 8.1.2.4. Let f_1 and f_2 in $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$. Let H be a morphism of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$ which is a zero cycle. Let x be an element of $\mathcal{A}(A_0, A_1)$. Since H is a cycle, we have the relation

$$m_1^{\mathcal{B}}(h_1(x)) - m_2^{\mathcal{B}}(h_{A_1} \odot f_1 x) + m_2^{\mathcal{B}}(f_2 x \odot h_{A_0}) = 0.$$

So we have a commutative diagram in $H^0\mathcal{B}$

$$\begin{tikzcd} \dot{f}_1 A_0 \arrow[r, "f_1 x"] \arrow[d, "h_{A_0}"'] & \dot{f}_1 A_1 \arrow[d, "h_{A_1}"] \\ f_2 A_0 \arrow[r, "f_2 x"'] & f_2 A_1. \end{tikzcd}$$

8.1.3 The A_∞-category $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$

Let us return to the notations of the section 8.1.1 but now suppose that $\mathcal{A}$ and $\mathcal{B}$ are strictly unital. The A_∞-category $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$ whose objects are the strictly unital A_∞-functors is defined as follows:

Let f_1 and f_2 be two A_∞-functors $\mathcal{A} \to \mathcal{B}$. An element H of $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$ is *strictly unital* if it satisfies

$$h_i(\mathbf{1}^{\odot\alpha} \odot \eta \odot \mathbf{1}^{\odot\beta}) = 0, \quad i \geq 1.$$

Strictly unital A_∞-functors and strictly unital morphisms of strictly unital A_∞-functors form a sub-A_∞-category of $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$. We denote it $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$. We verify that $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$ is functorial with respect to strictly unital A_∞-functors.

The 2-category cat_∞

Definition 8.1.3.1. Let cat_∞ be the *category* whose objects are the small strictly unital A_∞-categories, whose morphism spaces are the categories

$$\mathsf{cat}_\infty(\mathcal{A},\mathcal{B}) = H^0\mathsf{Func}_\infty(\mathcal{A},\mathcal{B}).$$

It follows from the functoriality of $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$ that cat_∞ is a 2-category.

Remark 8.1.3.2. Let f_1 and $f_2 \in \mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$. Let H be a morphism of $\mathsf{Hom}_{\mathsf{Func}_\infty}(f_1, f_2)$ which is a zero cycle. Let $\mathbf{I}_A$ be the identity morphism of $A \in \mathcal{A}$. Since H is a cycle, we have the relation

$$m_1^{\mathcal{B}}(h_1(\mathbf{I}_A)) - m_2^{\mathcal{B}}(h_A \odot f_1\mathbf{I}_A) + m_2^{\mathcal{B}}(f_2\mathbf{I}_A \odot h_A) =$$

$$-m_2^{\mathcal{B}}(h_A \odot \mathbf{I}_{f_1 A}) + m_2^{\mathcal{B}}(\mathbf{I}_{f_2 A} \odot h_A) = 0.$$

So we have a commutative diagram in $H^0\mathcal{B}$

$$\begin{tikzcd} \dot{f}_1 A \arrow[r, "\mathbf{I}"] \arrow[d, "h_A"'] & \dot{f}_1 A \arrow[d, "h_A"] \\ f_2 A \arrow[r, "\mathbf{I}"'] & f_2 A. \end{tikzcd}$$

8.2 The homotopy theory of A_∞-functors

This section is divided into two subsections. Let $\mathcal{A}$ and $\mathcal{B}$ be two strictly unital A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$. In the first, we construct a generalization of the A_∞-functor of Yoneda y (7.1.0.1): we define a A_∞-functor

$$z : \mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to \mathcal{C}_\infty(\mathcal{A},\mathcal{B}), \quad \mathcal{A},\mathcal{B} \in \mathsf{cat}_\infty,$$

which gives us Yoneda's A_∞-functor for $\mathcal{A}$ equal to $e_{\mathbb{B}}$. We then show that the generalized Yoneda A_∞-functor z induces a quasi-isomorphism in the spaces of morphisms. In the second part, we define the weak equivalences of the A_∞-category $\mathsf{Func}_\infty(\mathcal{A},\mathcal{B})$ (they are the A_∞-categorical analogue of the *homotopies* between A_∞-morphisms) and we characterize them using their images by the A_∞-function z.

8.2.1 The generalized Yoneda A_∞-functor

The generalized Yoneda A_∞-functor

$$z : \mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to \mathcal{C}_\infty(\mathcal{A},\mathcal{B})$$

is defined by the composition

$$\mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to \mathsf{Func}_\infty(\mathcal{A},\mathcal{C}_\infty\mathcal{B}) \xrightarrow{\theta^{-1}} \mathcal{C}_\infty(\mathcal{A},\mathcal{B})$$

where the first arrow is induced by the Yoneda functor $y : \mathcal{B} \to \mathcal{C}_\infty\mathcal{B}$ of chapter 7 and where θ is defined in the proposition below.

Proposition 8.2.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$. There is a functorial isomorphism of differential graded categories

$$\theta : \mathcal{N}_\infty(\mathcal{A},\mathcal{B}) \xrightarrow{\sim} \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}).$$

It restricts to an isomorphism

$$\theta : \mathcal{C}_\infty(\mathcal{A},\mathcal{B}) \xrightarrow{\sim} \mathsf{Func}_\infty(\mathcal{A},\mathcal{C}_\infty\mathcal{B})$$

if $\mathcal{A}$ and $\mathcal{B}$ are strictly unital.

Proof of Proposition (8.2.1.1) :

The functor θ

Recall (5.3.0.3) the map

$$\begin{array}{rcl} \mathsf{Obj}\,\mathcal{N}_\infty(\mathcal{A},\mathcal{B}) & \to & \mathsf{Obj}\,\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}), \\ M & \mapsto & \theta_M, \end{array}$$

is a bijection. We will extend this map to an isomorphism of differential graded categories

$$\theta : \mathcal{N}_\infty(\mathcal{A},\mathcal{B}) \to \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}).$$

Let X and X' be two $\mathcal{A}$-$\mathcal{B}$-bipolydules and

$$f : X \to X'$$

a morphism of $\mathcal{N}(\mathcal{A},\mathcal{B})$. It is given by morphisms

$$f_{i,j} : \mathcal{A}^{\odot i} \odot X \odot \mathcal{B}^{\odot j} \to X', \quad i, j \geq 0.$$

The morphism

$$\theta(f) \in \mathsf{Hom}_{\mathsf{Nunc}_\infty}(\theta_X, \theta_{X'})$$

is given by the morphism

$$B^+\mathcal{A} \to {}_{\theta_{X'}}\big(\mathcal{N}_\infty\mathcal{B}\big)_{\theta_X} = \mathsf{Hom}_{T^cS\mathcal{B}}(SX \odot T^cS\mathcal{B}, SX' \odot T^cS\mathcal{B})$$

which sends an element ϕ of $(S\mathcal{A})^{\odot i}$ of degree $|\phi|$ to the unique morphism (see 2.1.2.1) Υ such that the composition $p_1 \circ \Upsilon$ has as components the morphisms

$$SX \odot (S\mathcal{B})^{\odot j} \xrightarrow{(-1)^{|\phi|}\phi\odot\mathbf{1}} (S\mathcal{A})^{\odot i} \odot SX \odot (S\mathcal{B})^{\odot j} \xrightarrow{F_{i,j}} SX', \quad j \geq 0.$$

Note that if $i = 0$, the morphism

$$\Upsilon : SX \odot T^cS\mathcal{B} \to SX' \odot T^cS\mathcal{B}$$

is the morphism given by the morphisms $F_{0,j}$, $j \geq 0$. We have thus defined an isomorphism of graded objects

$$\mathsf{Hom}_{\mathcal{N}_\infty(\mathcal{A},\mathcal{B})}\big(X, X'\big) \to \mathsf{Hom}_{\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B})}\big(\theta_X, \theta_{X'}\big).$$

Let us show that this isomorphism defines an isomorphism of differential graded categories. Let f be of degree p. Compatibility with the composition m_2 is immediate. Let's show compatibility of m_1. Let $\phi \in (S\mathcal{A})^{\odot n}$ of degree $|\phi|$ and let $\kappa \odot \psi \in SX \odot (S\mathcal{B})^{\odot n'}$. We have the equalities (the calculation is the same as in the proof of the key lemma 5.3.0.1)

$$\begin{aligned}
&m_1^{\mathcal{B}}(\theta(f))(\phi)(\kappa\odot\psi)\\
&\quad = (-1)^{|\phi|+1}\Big[\sum b^{X'}_{0,\beta'}(f_{n,\beta}\odot\mathbf{1}^{\odot\beta'})(\phi\odot\kappa\odot\psi)\\
&\qquad\qquad - (-1)^p\sum f_{i,j}(\mathbf{1}^{\odot n}\odot b^X_{0,\alpha}\odot\mathbf{1}^{\odot\beta})(\phi\odot\kappa\odot\psi)\\
&\qquad\qquad - (-1)^p\sum f_{i,j}(\mathbf{1}^{\odot n}\odot\mathbf{1}_X\otimes\mathbf{1}^{\odot\alpha}\odot b^{\mathcal{B}}\odot\mathbf{1}^{\odot\beta})(\phi\odot\kappa\odot\psi)\\
&\qquad\qquad - (-1)^p\sum f_{i,j}(\mathbf{1}^{\odot\alpha}\odot b^{\mathcal{A}}\odot\mathbf{1}^{\odot\beta}\odot\mathbf{1}_X\otimes\mathbf{1}^{\odot n'})(\phi\odot\kappa\odot\psi)\Big],
\end{aligned}$$

$$\begin{aligned}
&- m_2^{\mathcal{B}}(\theta_{X'},\theta(f))(\phi)(\kappa\odot\psi)\\
&\quad = (-1)^{|\phi|+1}\sum_{\alpha'>0} b^{X'}_{\alpha',\beta'}(\mathbf{1}^{\odot\alpha'}\odot f_{\alpha,\beta}\odot\mathbf{1}^{\odot\beta'})(\phi\odot\kappa\odot\psi),
\end{aligned}$$

$$\begin{aligned}
&m_2^{\mathcal{B}}(\theta(f),\theta_X)(\phi)(\kappa\odot\psi)\\
&\quad = -(-1)^{p+|\phi|+1}\sum_{\alpha>0} f_{\alpha',\beta'}(\mathbf{1}^{\odot\alpha'}\odot b^X_{\alpha,\beta}\odot\mathbf{1}^{\odot\beta'})(\phi\odot\kappa\odot\psi).
\end{aligned}$$

We deduce the equality

$$d(\theta(f)) = m_1^{\mathcal{F}}(\theta(f)) - m_2^{\mathcal{F}}(\theta'_X \odot \theta(f)) + m_2^{\mathcal{F}}(\theta(f) \odot \theta_X) = \theta(d(f))$$

and we have the result.

Compatibility of θ with functoriality

If $f : \mathcal{A}' \to \mathcal{A}$ and $g : \mathcal{B} \to \mathcal{B}'$ are A$_\infty$-functors, they clearly induce morphisms which make the below squares commutative

$$\begin{tikzcd}
\mathcal{N}_\infty(\mathcal{A},\mathcal{B}) \arrow[r, "f^*"] \arrow[d, "\theta"'] & \mathcal{N}_\infty(\mathcal{A}',\mathcal{B}) \arrow[d, "\theta"] \\
\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}) \arrow[r, "f^*"'] & \mathsf{Nunc}_\infty(\mathcal{A}',\mathcal{N}_\infty\mathcal{B})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{N}_\infty(\mathcal{A},\mathcal{B}) \arrow[r, "g_*"] \arrow[d, "\theta"'] & \mathcal{N}_\infty(\mathcal{A},\mathcal{B}') \arrow[d, "\theta"] \\
\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}) \arrow[r, "g_*"'] & \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}').
\end{tikzcd}$$

The strictly unital case

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are strictly unital A$_\infty$-categories. We have the subcategories (5.2)

$$\mathcal{C}_\infty(\mathcal{A},\mathcal{B}) \subset \mathcal{N}_\infty(\mathcal{A},\mathcal{B}) \quad \text{and} \quad \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{C}_\infty\mathcal{B}) \subset \mathsf{Func}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}).$$

By remark (5.3.0.6), the bijection

$$\begin{array}{rcl} \mathsf{Obj}\,\mathcal{N}_\infty(\mathcal{A},\mathcal{B}) & \to & \mathsf{Obj}\,\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{N}_\infty\mathcal{B}), \\ M & \mapsto & \theta_M, \end{array}$$

is restricted to a bijection

$$\mathsf{Obj}\,\mathcal{C}_\infty(\mathcal{A},\mathcal{B}) \to \mathsf{Obj}\,\mathsf{Func}_\infty(\mathcal{A},\mathcal{C}_\infty\mathcal{B})$$

and it is clear that, for X and X' in $\mathcal{C}_\infty(\mathcal{A},\mathcal{B})$, the map $f \mapsto \theta(f)$ induces an isomorphism

$$\mathsf{Hom}_{\mathcal{C}_\infty(\mathcal{A},\mathcal{B})}\big(X, X'\big) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Func}_\infty(\mathcal{A},\mathcal{C}_\infty\mathcal{B})}\big(\theta_X, \theta_{X'}\big).$$

So we have an isomorphism of differential graded categories

$$\theta : \mathcal{C}_\infty(\mathcal{A},\mathcal{B}) \xrightarrow{\sim} \mathsf{Func}_\infty(\mathcal{A},\mathcal{C}_\infty\mathcal{B}).$$

$\square$

Theorem 8.2.1.2. The generalized Yoneda A$_\infty$-functor

$$z : \mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to \mathcal{C}_\infty(\mathcal{A},\mathcal{B})$$

induces a quasi-isomorphism on the morphism spaces.

Let's start with some lemmas.

Let $\big(\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})\big)_u$ be the full *subcategory* of $\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$ formed from the strictly unital A$_\infty$-functors .

Lemma 8.2.1.3. The faithful functor

$$\mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$$

induces an isomorphism

$$H^*\mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to H^*\big(\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})\big)_u.$$

Proof. In this proof, we use a filtration which is adapted from that of J. A. Guccione and J. J. Guccione [GG96].

Let f_1 and f_2 be two strictly unital A_∞-functors $\mathcal{A}\to\mathcal{B}$. We recall that the space of strictly unital elements of

$$\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1,f_2) = \mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}(T^cS\mathcal{A},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1})$$

is formed by the H which are factorized by $T^cS\overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is the cokernel of the unit of $\mathcal{A}$. So we have the equality

$$\mathsf{Hom}_{\mathsf{Func}_\infty}(f_1,f_2) = \mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}\Big(\bigoplus_{0\le p}(S\overline{\mathcal{A}})^{\odot p},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\Big).$$

For all $p\ge 0$, we set

$$F_p = \mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}\Big(\bigoplus_{0\le i<p}(S\overline{\mathcal{A}})^{\odot i},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\Big) \oplus \mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}\Big(\bigoplus_{0\le j}(S\overline{\mathcal{A}})^{\odot p}\odot(S\mathcal{A})^{\odot j},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\Big).$$

We clearly have the inclusion $F_{i+1}\subset F_i$, $i\ge 0$. The inverse limit of F_p, $p\ge 0$, is the space $\mathsf{Hom}_{\mathsf{Func}_\infty}(f_1,f_2)$ and F_0 is the space $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1,f_2)$. We have an injection of graded spaces

$$J_p : F_p \hookrightarrow \mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}(T^cS\mathcal{A},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}),\quad p\ge 0.$$

Provide $\mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}(T^cS\mathcal{A},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1})$ with the differential $m_1^{\mathsf{Nunc}_\infty}$ and show that it induces a differential on F_p, $p\ge 1$.

Let $p\ge 1$. Let Q_p be the projection onto the cokernel of J_p. Let $H\in\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1,f_2)$ such that $Q_p(H)=0$. This condition is equivalent to the fact that the morphisms h_i, $i\ge 0$, (defined in 8.1.1.2) satisfy the equations

$$h_i((\mathbf{1}^{\odot\alpha}\odot\eta\odot\mathbf{1}^{\odot\beta})\odot\mathbf{1}^{\odot\gamma})=0,\quad \alpha+1+\beta+\gamma=i,\quad \alpha+1+\beta\le p.$$

We deduce from the concrete description (8.1.1.8) of the element $m_1^{\mathsf{Nunc}_\infty}(H)$ that the composition of $m_1^{\mathsf{Nunc}_\infty}(H)$ with

$$((\mathbf{1}^{\odot\alpha}\odot\eta\odot\mathbf{1}^{\odot\beta})\odot\mathbf{1}^{\odot\gamma}),\quad \alpha+1+\beta+\gamma=i,\quad \alpha+1+\beta\le p,$$

cancels out. This shows that $Q_p(m_1^{\mathsf{Nunc}_\infty}(H))=0$. We deduce that the differential $m_1^{\mathsf{Nunc}_\infty}$ induces a differential on the graded object F_p, $p\ge 1$.

Let us show that the quotient of the inclusion $F_{p+1}\subset F_p$, $p\ge 0$, is contractible. Let G_p be the cokernel of this inclusion. It is isomorphic to

$$\mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}\Big(\bigoplus_{0\le j}(Se)^{\otimes p}\odot(S\mathcal{A})^{\odot j},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\Big) = \mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}\Big((Se)^{\odot p}\odot T^cS\mathcal{A},\, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\Big)$$

Let H be an element of F_i of degree $|H|$. We deduce from the concrete description (8.1.1.8) of $m_1^{\mathsf{Nunc}_\infty}(H)(\phi)$, where ϕ is an element of $(Se)^{\odot p}\odot T^cS\mathcal{A}$, equality

$$m_1^{G_p}(H) = m_1^{\mathcal{F}(\mathcal{A},\mathcal{B})}(H),$$

where $\mathcal{F}(\mathcal{A},\mathcal{B})$ is the category equipped with naive compositions (8.1.1). By definition, the element $m_1^{\mathcal{F}(\mathcal{A},\mathcal{B})}(H)$ is equal to

$$b^{B^+\mathcal{A}} \circ H - (-1)^{|H|} H \circ m_1^{\mathcal{B}}.$$

As the A_∞-category $\mathcal{A}$ is strictly unital, it is H-unital (4.1.3.7). Its bar construction is therefore quasi-isomorphic to 0. We deduce that G_p is contractile.

Let us show that the inclusion

$$J : \mathsf{Hom}_{\mathsf{Func}_\infty}(f_1, f_2) \hookrightarrow \mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$$

is a quasi-isomorphism. The complexes G_p, $p \geq 0$, are all contractible. We deduce that the cokernel of the injection J_p, $p \geq 0$, is isomorphic to

$$\bigoplus_{0 \leq i \leq p} G_i.$$

It is a contractible space. The space $\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f_1, f_2)$ is therefore isomorphic to

$$F_p \oplus \bigoplus_{0 \leq i \leq p} G_i, \quad p \geq 0.$$

The cokernel of J is therefore

$$\prod_{0 \leq i} G_i.$$

It is clearly contractible, hence the result. □

Lemma 8.2.1.4. Let $\mathcal{A}'$ and $\mathcal{B}'$ be A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$ and let

$$g : \mathcal{A} \to \mathcal{A}' \quad \text{and} \quad g' : \mathcal{B} \to \mathcal{B}'$$

be A_∞-quasi-isomorphisms in $\mathsf{C}(\mathbb{A},\mathbb{A})$ and $\mathsf{C}(\mathbb{B},\mathbb{B})$. Consider them as A_∞-functors (5.1.2.7). The A_∞-functors

$$g^* : \mathsf{Nunc}_\infty(\mathcal{A}',\mathcal{B}) \to \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B}) \quad \text{and} \quad g'_* : \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B}) \to \mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B}')$$

induce quasi-isomorphisms in the spaces of morphisms.

We deduce from this lemma and from lemma (8.2.1.3) the following corollary:

Corollary 8.2.1.5. Reuse the hypotheses of lemma (8.2.1.4). If the A_∞-categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ and $\mathcal{B}'$ are strictly unital and the A_∞-morphisms g and g' are strictly unital, the restricted A_∞-functors

$$\mathsf{Func}_\infty(\mathcal{A}',\mathcal{B}) \to \mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \quad \text{and} \quad \mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \to \mathsf{Func}_\infty(\mathcal{A},\mathcal{B}')$$

induce quasi-isomorphisms in the spaces of morphisms. □

Proof of Lemma 8.2.1.4: By the proposition (6.1.3.4), it suffices to show that the A_∞-functors induced by g and g'

$$\mathcal{F}(\mathcal{A}',\mathcal{B}) \to \mathcal{F}(\mathcal{A},\mathcal{B}) \quad \text{and} \quad \mathcal{F}(\mathcal{A},\mathcal{B}) \to \mathcal{F}(\mathcal{A},\mathcal{B}'),$$

where $\mathcal{F}(\mathcal{A}',\mathcal{B})$, $\mathcal{F}(\mathcal{A},\mathcal{B})$, $\mathcal{F}(\mathcal{A},\mathcal{B})$ and $\mathcal{F}(\mathcal{A},\mathcal{B}')$ are the categories endowed with naive compositions (see 8.1.1), give quasi-isomorphisms in the spaces of morphisms. The morphism spaces

$$\mathsf{Hom}_{\mathcal{F}(\mathcal{A},\mathcal{B})}(f_1,f_2)=\mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}\big(T^cS\mathcal{A},{}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}\big)$$

are equipped with the differential

$$\delta : H\mapsto m_1^{\mathcal{B}}\circ H-(-1)^{|H|}H\circ b^{B^+\mathcal{A}}.$$

As the morphisms $g_1' : \mathcal{B}\to\mathcal{B}'$ and $B^+g : B^+\mathcal{A}\to B^+\mathcal{A}'$ are quasi-isomorphisms, we have the result. $\square$

Proof of Theorem (8.2.1.2): We will first show that we can reduce to the case where the strictly unital A_∞-categories are unital differential graded, then we will prove the result using arguments from classical homological algebra.

The proposition (7.5.0.2) gives us unital differential graded models $\mathcal{A}'$ and $\mathcal{B}'$ equipped with strictly unital A_∞-quasi-isomorphisms

$$\mathcal{A}\to\mathcal{A}'\quad\text{and}\quad\mathcal{B}\to\mathcal{B}'.$$

The lemma 8.2.1.4 and its corollary 8.2.1.5 gives us a diagram

$$\begin{tikzcd}
\mathsf{Func}_\infty(\mathcal{A},\mathcal{B}) \arrow[r, "z"] \arrow[d] & \mathcal{C}_\infty(\mathcal{A},\mathcal{B}) \arrow[d] \\
\mathsf{Func}_\infty(\mathcal{A},\mathcal{B}') \arrow[r, "z"] & \mathcal{C}_\infty(\mathcal{A},\mathcal{B}') \\
\mathsf{Func}_\infty(\mathcal{A}',\mathcal{B}') \arrow[r, "z"] \arrow[u] & \mathcal{C}_\infty(\mathcal{A}',\mathcal{B}') \arrow[u]
\end{tikzcd}$$

of which all the vertical arrows induce quasi-isomorphisms in the morphism spaces. It is therefore enough for us to show that

$$z : \mathsf{Func}_\infty(\mathcal{A},\mathcal{B})\to\mathcal{C}_\infty(\mathcal{A},\mathcal{B})$$

is a quasi-isomorphism in the case where $\mathcal{A}$ and $\mathcal{B}$ are unital differential graded. The lemma (8.2.1.4) and the proposition (5.2.0.4) show that it is equivalent to show that

$$z : \big(\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})\big)_u\to\big(\mathcal{N}_\infty(\mathcal{A},\mathcal{B})\big)_u$$

is a quasi-isomorphism. Let f_1 and f_2 be strictly unital A_∞-functors $\mathcal{A}\to\mathcal{B}$. We have an isomorphism

$$\mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{A})}(B^+\mathcal{A},{}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1})\longrightarrow\mathsf{Hom}_{\mathcal{A}^{op}\odot\mathcal{A}}(\mathcal{A}\odot B^+\mathcal{A}\odot\mathcal{A},{}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1}).$$

Recall (7.4.0.1) that the A_∞-functor of Yoneda

$$y : \mathcal{B}\to\mathcal{C}_\infty\mathcal{B}$$

induces a quasi-isomorphism in the spaces of morphisms. So we have a quasi-isomorphism

$$\begin{array}{c}\mathsf{Hom}_{\mathcal{A}^{op}\odot\mathcal{A}}(\mathcal{A}\odot B^{+}\mathcal{A}\odot\mathcal{A}, {}_{\dot{f}_2}\mathcal{B}_{\dot{f}_1})\\ \downarrow \\ \mathsf{Hom}_{\mathcal{A}^{op}\odot\mathcal{A}}\big(\mathcal{A}\odot B^{+}\mathcal{A}\odot\mathcal{A}, \mathsf{Hom}_{\mathcal{C}_\infty\mathcal{B}}(y\circ f_1, y\circ f_2)\big).\end{array}$$

Let $\mathbf{R}\mathsf{Hom}$ be the right derived functor that calculates the groups Ext^*. The last term above can be rewritten

$$\mathbf{R}\mathsf{Hom}_{\mathcal{A}^{op}\odot\mathcal{A}}\big(\mathcal{A}, \mathbf{R}\mathsf{Hom}_{\mathcal{B}}(y\circ f_1, y\circ f_2)\big).$$

It is isomorphic to

$$\mathbf{R}\mathsf{Hom}_{\mathcal{A}^{op}\odot\mathcal{B}}\big(y\circ f_1, y\circ f_2\big)$$

which is isomorphic to

$$\mathsf{Hom}_{\mathcal{A}^{op}\odot\mathcal{B}}\big(\mathcal{A}\odot T^c S\mathcal{A}\odot S(y\circ f_1)\odot T^c S\mathcal{B}\odot\mathcal{B}, S(y\circ f_2)\big)\simeq$$

$$\mathsf{Hom}_{\mathsf{C}(\mathbb{A},\mathbb{B})}\big(T^c S\mathcal{A}\odot S(y\circ f_1)\odot T^c S\mathcal{B}, S(y\circ f_2)\big)\simeq$$

$$\mathsf{Hom}_{(T^c S\mathcal{A})^{op}\odot(T^c S\mathcal{B})}\big(B(y\circ f_1), B(y\circ f_2)\big).$$

As we have equalities of $\mathcal{A}$-$\mathcal{B}$-bipolydules

$$y\circ f = z(f),\quad f\in\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B}),$$

the lemma (8.2.1.1) shows that the last space of morphisms above is

$$\mathsf{Hom}_{\mathcal{N}_\infty(\mathcal{A},\mathcal{B})}\big(z(f_1), z(f_2)\big).$$

The composition of all the (quasi-)isomorphisms above being the morphism

$$z(f_1,f_2) : \mathsf{Hom}_{\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})}(f_1,f_2)\to\mathsf{Hom}_{\mathcal{N}_\infty(\mathcal{A},\mathcal{B})}(f_1,f_2),$$

we have the result. □

Remark 8.2.1.6. By construction, the image of the A$_\infty$-functor z is made up of the $\mathcal{A}$-$\mathcal{B}$-bipolydules which are of the form

$$\mathcal{B}(?, \dot{f}-),\quad f\in\mathsf{Func}_\infty(\mathcal{A},\mathcal{B}).$$

They are free as $\mathcal{B}$-polydules.

8.2.2 Weak equivalences of A$_\infty$-functors

Weak equivalences between A$_\infty$-functors are the A$_\infty$-categorical analogue of *homotopies* between A$_\infty$-morphisms.

Definition 8.2.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two A$_\infty$-categories over $\mathbb{A}$ and $\mathbb{B}$. Let f and g be two A$_\infty$-functors $\mathcal{A}\to\mathcal{B}$. An element $H\in Z^0\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f,g)$ is a *weak equivalence* if it becomes an isomorphism in $H^0\mathsf{Nunc}_\infty(\mathcal{A},\mathcal{B})$. We will then say that f and g are *weakly equivalent* and write $f\sim g$.

Remark 8.2.2.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are strictly unital and f and g are strictly unital A_∞-functors. According to lemma (8.2.1.3) f and g are weakly equivalent if and only if there exists a strictly unital morphism $H \in Z^0\mathsf{Hom}_{\mathsf{Func}_\infty}(f_1, f_2)$ which becomes an isomorphism in $H^0\mathsf{Func}_\infty(\mathcal{A}, \mathcal{B})$.

Proposition 8.2.2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two strictly unital A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$. Let f and g be two strictly unital A_∞-functors $\mathcal{A} \to \mathcal{B}$. An element $H \in Z^0\mathsf{Hom}_{\mathsf{Nunc}_\infty}(f, g)$ is a weak equivalence if and only if $h_0 : e_{\mathbb{A}} \to \mathsf{Hom}_{\mathcal{B}}(\dot{f}?, \dot{g}—)$ induces an isomorphism of functors $H^0 f \to H^0 g$ from $H^0\mathcal{A}$ into $H^0\mathcal{B}$.

Proof. According to theorem (8.2.1.2), we have an isomorphism

$$H^0\mathsf{Func}_\infty(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} H^0\mathcal{C}_\infty(\mathcal{A}, \mathcal{B}).$$

The element H is therefore a weak equivalence if and only if the morphism of $\mathcal{A}$-$\mathcal{B}$-bipolydules

$$z(H) : z(f) \to z(g)$$

is a homotopy equivalence in $\mathcal{C}_\infty(\mathcal{A}, \mathcal{B})$, i.e. (see the equivalence between D2 and D3 in 4.1.3.1) if and only if $z(H)$ is an A_∞-quasi-isomorphism of $\mathcal{A}$-$\mathcal{B}$-bipolydules. By the definition of A_∞-quasi-isomorphisms, this is equivalent to the fact that the morphism of $\mathsf{C}(\mathbb{A}, \mathbb{B})$

$$S^{-1}(z(H))_{0,0} : \mathcal{B}(?, f—) \to \mathcal{B}(?, g—)$$

is a quasi-isomorphism, that is, it becomes an isomorphism in cohomology. As Yoneda's functor in the classical sense sends the class in $H^*\mathcal{B}$ of

$$h_A = h_0(\mathbf{I}_A) : \dot{f}A \to \dot{g}A, \quad A \in \mathbb{A},$$

to

$$S^{-1}(z(H))_{0,0} : H^*\mathcal{B}(?, fA) \to H^*\mathcal{B}(?, gA),$$

$S^{-1}(z(H))_{0,0}$ is a quasi-isomorphism if and only if h_A induces an isomorphism in $H^*\mathcal{B}$, or equivalently in $H^0\mathcal{B}$. $\square$

Chapter 9

A_∞-equivalences

This chapter is divided into two parts. In the section 9.1, we define the A_∞-isomorphism of an A_∞-category $\mathcal{A}$ and we will show that this notion is an A_∞-categorical lifting of the isomorphism of $H^0\mathcal{A}$ in the classical sense. In the section 9.2, we define A_∞-equivalences and we show that an A_∞-functor f is an A_∞-equivalence if and only if f_1 is a quasi-isomorphism and $H^0 f_1 : H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence of categories in the classical sense. This characterization of A_∞-equivalences was stated by M. Kontsevich [Kon98]. K. Fukaya demonstrated this independently [Fuk01b, thm. 8.6], as well as V. Lyubashenko [Lyu02].

9.1 A_∞-isomorphism

Let $\mathbb{O}$ be a set. Consider a category as follows: the objects are in bijection with $\mathbb{O}$ and, for $i, j \in O$, the space of morphisms $\mathsf{Hom}_{\mathbb{O}}(i,j)$ contains a unique element denoted (i,j). The composition is then necessarily given by

$$(j,k) \circ (i,j) = (i,k), \quad i,j,k \in \mathbb{O}.$$

In particular, the identity of $i \in \mathbb{O}$ is the morphism (i,i) and all the morphisms (i,j) are isomorphisms.

Definition 9.1.0.1. Let $n \geq 1$. Consider the set of n elements $\{1, \ldots, n\}$. Let $\mathbf{I}_n$ be a $\mathbb{K}$-category generated by the category $\{1, \ldots, n\}$.

Remark 9.1.0.2. Let $n \geq 2$. Let $\mathcal{A}$ be a $\mathbb{K}$-category and objects $A_i \in \mathsf{Obj}\,\mathcal{A}$, $1 \leq i \leq n$. They are isomorphic if and only if there is a functor

$$f : \mathbf{I}_n \to \mathcal{A}$$

which sends i to A_i. We then say that f is an *isomorphism functor* for the objects $A_i \in \mathsf{Obj}\,\mathcal{A}$, $1 \leq i \leq n$.

Definition 9.1.0.3. Let $n \geq 2$. Let $\mathcal{A}$ be a strictly unital A_∞-category over $\mathbb{A}$ and objects $A_i \in \mathbb{A}$, $1 \leq i \leq n$. The objects $A_i \in \mathbb{A}$, $1 \leq i \leq n$, are A_∞*-isomorphic* if there exists a strictly unital A_∞-functor

$$f : \mathbf{I}_n \to \mathcal{A}$$

which sends i to A_i. We then say that f is an A_∞*-isomorphic* A_∞*-functor* for the objects $A_i \in \mathbb{A}$, $1 \leq i \leq n$.

We now prove a lemma stated in [Kon98]:

Lemma 9.1.0.4. Let $\mathcal{A}$ be a strictly unital A_∞-category. Let $n \geq 1$. Objects $A_i \in \mathbb{A}$, $1 \leq i \leq n$, are A_∞-isomorphic in $\mathcal{A}$ if and only they are isomorphic in $H^0\mathcal{A}$.

Proof. As $\mathcal{A}$ is strictly unital, there exists (3.2.4.1) a strictly unital minimal model $H^*\mathcal{A}$ for $\mathcal{A}$ and strictly unital A_∞-functors (3.2.4.1)

$$i : H^*\mathcal{A} \to \mathcal{A} \quad \text{and} \quad q : \mathcal{A} \to H^*\mathcal{A}.$$

We deduce that objects $A_i \in \mathbb{A}$, $1 \leq i \leq n$, are A_∞-isomorphic in $\mathcal{A}$ if and only if they are A_∞-isomorphic in $H^*\mathcal{A}$. We can therefore assume that the A_∞-category $\mathcal{A}$ is minimal.

Let $\mathcal{A}$ be a minimal A_∞-category. Let us show that the A_∞-isomorphism in $\mathcal{A}$ leads to the isomorphism in $H^0\mathcal{A}$. Let $f : \mathbf{I}_n \to \mathcal{A}$ be an A_∞-isomorphism A_∞-functor for $A_i \in \mathbb{A}$, $1 \leq i \leq n$. Since the A_∞-categories $\mathbf{I}_n$ and $\mathcal{A}$ are minimal, $f_0 : \mathbf{I}_n \to \mathcal{A}^0 = H^0\mathcal{A}$ defines an isomorphism functor for the objects $A_i \in \mathbb{A}$, $1 \leq i \leq n$.

Let us show that the isomorphism in $H^0\mathcal{A}$ implies the A_∞-isomorphism in $\mathcal{A}$. Let $g : \mathbf{I}_n \to H^0\mathcal{A}$ be an isomorphism functor for the objects $A_i \in \mathsf{Obj}\, H^0\mathcal{A}$, $1 \leq i \leq n$. We are looking for a strictly unital A_∞-functor

$$f : \mathbf{I}_n \to \mathcal{A}$$

such that $f_1 = i \circ g$, where i is the inclusion $\mathcal{A}^0 \hookrightarrow \mathcal{A}$. According to the theorem (3.2.2.1), it suffices to construct an A_∞-functor f' (not necessarily strictly unital) such that $f'_1 = f_1$. We are going to construct the f'_r, $r \geq 2$, by induction on r. Suppose given graded morphisms f'_i, $1 \leq i \leq r$, of degree $1 - i$, defining a A_r-functor $\mathbf{I}_n \to \mathcal{A}$. Let f'_{r+1} be a morphism of degree $-r$. The lemma (B.4.2) asserts that the sequence of f'_i, $1 \leq i \leq r+1$, defines an $\mathrm{A}_{\mathrm{r+1}}$-functor if we we have equality

$$\delta_{Hoch}(f'_{r+1}) = -r(f'_2, \ldots, f'_r)$$

where $r(f'_2, \ldots, f'_r)$ is some cycle of the Hochschild complex $C^*(\mathbf{I}_n, {}_{\dot{f}'}\mathcal{A}_{\dot{f}'})$. As the category $\mathbf{I}_n$ is equivalent to the trivial category $\mathbf{I}_1$, the Hochschild complex $C^*(\mathbf{I}_n, {}_{\dot{f}}\mathcal{A}_{\dot{f}})$ is acyclic. There thus exists a morphism f'_{r+1} such that the graded morphisms f'_i, $1 \leq i \leq r+1$, define an $\mathrm{A}_{\mathrm{r+1}}$-function $\mathbf{I}_n \to \mathcal{A}$. $\square$

9.2 The characterization of A_∞-equivalences

Definition 9.2.0.1. Two strictly unital A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{A}$ and $\mathbb{B}$ are A_∞*-equivalent* if there exist strictly unital A_∞-functors

$$f : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad g : \mathcal{B} \to \mathcal{A}$$

such that $f \circ g$ and $\mathbf{1}_\mathcal{B}$ are A_∞-isomorphic in $\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$ and $g \circ f$ and $\mathbf{1}_\mathcal{A}$ are A_∞-isomorphic in $\mathsf{Func}_\infty(\mathcal{A}, \mathcal{A})$. We will then say that f (or g) is an A_∞*-equivalence* between $\mathcal{A}$ and $\mathcal{B}$.

Definition 9.2.0.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital differential graded categories over $\mathbb{A}$ and $\mathbb{B}$. They are *equivalent* (in the classical sense) if there exist functors

$$f : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad g : \mathcal{B} \to \mathcal{A}$$

and isomorphisms of functors

$$\mu : f \circ g \to \mathbf{1}_\mathcal{B} \quad \text{and} \quad \nu : g \circ f \to \mathbf{1}_\mathcal{A}.$$

We will then say that f (or g) is an *equivalence* between $\mathcal{A}$ and $\mathcal{B}$.

Remark 9.2.0.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital differential graded categories over $\mathbb{A}$ and $\mathbb{B}$. Suppose they are equivalent. Let f be an equivalence between $\mathcal{A}$ and $\mathcal{B}$. Let g, μ and ν as in the definition (9.2.0.2). The element $H \in \mathsf{Hom}_{\mathsf{Func}_\infty}(f \circ g, \mathbf{1}_{\mathcal{B}})$ (resp. $H' \in \mathsf{Hom}_{\mathsf{Func}_\infty}(g \circ f, \mathbf{1}_{\mathcal{A}})$) defined by

$$h_0 = \mu, \quad h_i = 0, \quad i \geq 1, \quad \Big(\text{resp.} \quad h'_0 = \mu, \quad h'_i = 0, \quad i \geq 1\Big)$$

is a cycle in $\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$ (resp. in $\mathsf{Func}_\infty(\mathcal{A}, \mathcal{A})$). It induces an isomorphism in $H^0\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$ (resp. $H^0\mathsf{Func}_\infty(\mathcal{A}, \mathcal{A})$). This shows that $\mathcal{A}$ and $\mathcal{B}$ are A_∞-equivalent as A_∞-categories.

The statement of the following theorem is due to M. Kontsevich [Kon98].

Theorem 9.2.0.4 (See also K. Fukaya [Fuk01b] and V. Lyubashenko [Lyu02])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two strictly unital A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$ and $f : \mathcal{A} \to \mathcal{B}$ a strictly unital A_∞-functor. The following statements are equivalent:

a. f is an A_∞-equivalence.

b. f_1 induces an equivalence $H^*\mathcal{A} \to H^*\mathcal{B}$, where $H^*\mathcal{A}$ and $H^*\mathcal{B}$ are the cohomology of $\mathcal{A}$ and $\mathcal{B}$ considered as graded $\mathbb{K}$-categories.

c. f_1 is a quasi-isomorphism and induces an equivalence $H^0\mathcal{A} \to H^0\mathcal{B}$.

Proof. $a \Rightarrow b$: Suppose that f is an A_∞-equivalence. Let $g : \mathcal{B} \to \mathcal{A}$ satisfying the conditions of definition (9.2.0.1). According to lemma (9.1.0.4), the A_∞-isomorphism in $\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$ (resp. in $\mathsf{Func}_\infty(\mathcal{A}, \mathcal{A})$) is equivalent to the isomorphism in $H^0\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$ (resp. in $H^0\mathsf{Func}_\infty(\mathcal{A}, \mathcal{A})$). As $f \circ g$ and $\mathbf{1}_{\mathcal{B}}$ are isomorphic in $H^0\mathsf{Func}_\infty(\mathcal{A}, \mathcal{A})$, there exists an element

$$H \in Z^0\mathsf{Hom}_{\mathsf{Func}_\infty}(g \circ f, \mathbf{1}_{\mathcal{B}})$$

inducing an isomorphism in $H^0\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$. According to proposition (8.2.2.3), the morphism h_0 induces an isomorphism of functors

$$H^0(h_0) : H^*(g_1 \circ f_1) \to H^*\mathbf{1}_{\mathcal{B}}.$$

The isomorphism of functors between $H^*(f_1 \circ g_1)$ and $\mathbf{1}_{H^*\mathcal{A}}$ is constructed in the same way.

$b \Rightarrow c$: This is clear.

$c \Rightarrow a$: We will show this implication in two particular cases and then we will show that it implies the general case.

First case: the map $\dot{f} : \mathbb{A} \to \mathbb{B}$ is a bijection.
We can consider that $\mathbb{A}$ is equal to $\mathbb{B}$ and that $\dot{f}$ is the identity of $\mathbb{A}$. The A_∞-functor f is thus (5.1.2.7) a A_∞-morphism in the category $\mathsf{C}(\mathbb{A}, \mathbb{A})$. According to point b of corollary (1.3.1.3), there exists an A_∞-morphism $g : \mathcal{B} \to \mathcal{A}$ and homotopies h and h' from $f \circ g$ to $\mathbf{1}_{\mathcal{B}}$ and from $g \circ f$ to $\mathbf{1}_{\mathcal{A}}$. Thanks to proposition (3.2.4.3), we can assume that the A_∞-morphism g and the homotopies h and h' are strictly unital. Let H be the element of $\mathsf{Hom}_{\mathsf{Func}_\infty}(f \circ g, \mathbf{1}_{\mathcal{B}})$ given (see 8.1.1.2) by the morphisms h_i, $i \geq 1$, and $h_A = \mathbf{1}_A$, $A \in \mathbb{A}$. Let $Z = m_1^{\mathsf{Func}_\infty}(H)$. It is given by morphisms z_i, $i \geq 0$. Let us show that H is a cycle in $\mathsf{Hom}_{\mathsf{Func}_\infty}(\mathcal{A}, \mathcal{B})$. The morphism z_0 is clearly zero. For $n \geq 1$, we check (using the fact that $f \circ g$, $\mathbf{1}_{\mathcal{B}}$ and h are strictly unital) that

$$\begin{aligned} z_n = \quad & (f \circ g)_n - (\mathbf{1}_{\mathcal{B}})_n \\ & - \sum (-1)^s m_{r+1+t}((f \circ g)_{i_1} \otimes \ldots \otimes (f \circ g)_{i_r} \otimes h_k \otimes (\mathbf{1}_{\mathcal{B}})_{j_1} \otimes \ldots \otimes (\mathbf{1}_{\mathcal{B}})_{i_t}) \\ & - \sum (-1)^{jk+l} h_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}). \end{aligned}$$

where s is the sign involved in the equation $(***_n)$ of (1.2.1.7). As h is a homotopy between $f \circ g$ and $\mathbf{1}_{\mathcal{B}}$, the term on the right is zero. This shows that H is a cycle of $\mathsf{Hom}_{\mathsf{Func}_\infty}(f \circ g, \mathbf{1}_{\mathcal{B}})$. The morphism h_A, $A \in \mathbb{A}$ being equal to $\mathbf{1}_A$, the proposition (8.2.2.3) implies that H induces an isomorphism in $H^0\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$. We deduce (9.1.0.4) that the A_∞-functors $\mathbf{1}_{\mathcal{B}}$ and $f \circ g$ are A_∞-isomorphic in $\mathsf{Func}_\infty(\mathcal{B}, \mathcal{B})$. The A_∞-isomorphism between $g \circ f$ and $\mathbf{1}_{\mathcal{A}}$ is shown in the same way.

Remark 9.2.0.5. In particular, this implies that a strictly unital A_∞-category $\mathcal{A}$ is A_∞-equivalent to its minimal model (3.2.4.1) and all its differential graded models (7.5.0.2).

Second case: f is an inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ where $\mathcal{A}$ is a full sub-A_∞-category of $\mathcal{B}$. Thanks to the previous remark, we can assume that $\mathcal{B}$ is differential graded. As $H^0 f : H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence, it suffices to prove the theorem in the following case: Let us choose in each isomorphism class $[B]$ of $\mathcal{B}$ a representative B_0. Let $\mathbb{A}$ be the set of these representatives. We set $\mathcal{A}$ equal to the full subcategory of $\mathcal{B}$ formed by the objects $A \in \mathbb{A}$. That is

$$r : \mathbb{B} \to \mathbb{A}, \quad B \mapsto r(B) = B_0.$$

Let $\mathcal{A}'$ be the differential graded category ${}_r\mathcal{A}_r$ over $\mathbb{B}$ (see 5.1.2.4). We then have the equalities

$$\mathcal{A}'(B, B') = \mathcal{A}(B_0, B'_0) = \mathcal{B}(B_0, B'_0)$$

and the differential graded categories $\mathcal{A}$ and $\mathcal{A}'$ are equivalent in the classical sense. It suffices therefore to show that $\mathcal{A}'$ and $\mathcal{B}$ are A_∞-equivalent. Let $i : \mathcal{A} \to \mathcal{A}'$ be the inclusion. We will construct an A_∞-equivalence

$$g : \mathcal{A}' \to \mathcal{B}$$

such that $f = g \circ i$.

Construction of g: Let $\dot{g} = \mathbf{1}_{\mathbb{B}}$. The A_∞-functor g is thus given by an A_∞-morphism $\mathcal{A}' \to \mathcal{B}$ in $\mathsf{C}(\mathbb{B}, \mathbb{B})$. By assumption, every element $B \in \mathbb{B}$ is A_∞-isomorphic to $r(B)$. For each $B \in \mathbb{B}$, choose an element α_B of $\mathcal{B}(r(B), B)$ that becomes an isomorphism in $H^0\mathcal{B}(r(B), B)$. Consider the diagram of differential graded $\mathbb{B}$-$\mathbb{B}$-bimodules

$$\text{(I)} \qquad \begin{tikzcd} & \mathcal{B}(-, ?) \arrow[d, "\alpha^*"] \\ \mathcal{A}'(-, ?) = \mathcal{B}(r(-), r(?)) \arrow[r, "\alpha_*"'] & \mathcal{B}(r(-), ?). \end{tikzcd}$$

The Yoneda A_∞-functor $y : \mathcal{B} \to \mathcal{C}_\infty\mathcal{B}$ (7.1.0.1) sends the diagram (I) to a quasi-isomorphic diagram of $\mathbb{B}$-$\mathbb{B}$-bimodules

$$\text{(I}'\text{)} \qquad \begin{tikzcd} & \mathcal{C}_\infty\mathcal{B}(y-, y?) \arrow[d, "(y\alpha)^*"] \\ \mathcal{C}_\infty\mathcal{B}(yr(-), yr(?)) \arrow[r, "(y\alpha)_*"'] & \mathcal{C}_\infty\mathcal{B}(yr(-), y?). \end{tikzcd}$$

For each $B \in \mathbb{B}$, the morphism α_B becomes an isomorphism in $H^0\mathcal{B}$. As the Yoneda functor induces a quasi-isomorphism in spaces of morphisms (7.4.0.1), it induces an isomorphism $H^0\mathcal{B} \to$

$H^0\mathcal{C}_\infty\mathcal{B}$. We deduce that the morphism $y\alpha_B$ is a homotopy equivalence in $\mathcal{C}_\infty\mathcal{B}$. According to the equivalence between the categories of D3 and D4 (4.1.3.1), it is a quasi-isomorphism. This implies that the arrows of the diagram (I') are quasi-isomorphisms. The category $\mathcal{B}$ being differential graded, the $\mathcal{B}$-bipolydules $y(B)$, $B \in \mathbb{B}$, are differential graded $\mathcal{B}$-modules and the morphism $y\alpha_B : y(r(B)) \to y(B)$ is a morphism of differential graded $\mathcal{B}$-modules. The axiom (CM5) of the category $\mathsf{Mod}\,\mathcal{B}$ gives us a factorization of $y\alpha_B$ into a trivial cofibration and a trivial fibration

$$yr(B) \overset{i_B}{\rightarrowtail} m(B) \overset{p_B}{\twoheadrightarrow} yB.$$

Thanks to the axiom (CM4) of the category $\mathsf{Mod}\,\mathcal{B}$, there exists a quasi-isomorphism σ_B such that $p_B \circ \sigma_B = \mathbf{1}_{yB}$. The morphism

$$\mathcal{C}_\infty\mathcal{B}(y-, y?) \to \mathcal{C}_\infty\mathcal{B}(m-, m?), \quad x \mapsto \sigma \circ x \circ p,$$

is a quasi-isomorphism of differential graded algebras. The diagram

$$(\mathrm{I}'') \qquad \begin{tikzcd} & \mathcal{C}_\infty\mathcal{B}(m-, m?) \arrow[d, two heads, "i^*"] \\ \mathcal{C}_\infty\mathcal{B}(yr(-), yr(?)) \arrow[r, "(y\alpha)_*"'] & \mathcal{C}_\infty\mathcal{B}(yr(-), y?) \end{tikzcd}$$

is thus quasi-isomorphic to (I'). The cofibrations being homomorphisms, the vertical arrow of the diagram (I'') is a surjection. We deduce that the canonical projections

$$\mathcal{C}_\infty\mathcal{B}(yr(-), yr(?)) \leftarrow P \to \mathcal{C}_\infty\mathcal{B}(m-, m?),$$

where P is the pullback above the diagram (I'') are quasi-isomorphisms. Since $\mathcal{C}_\infty\mathcal{B}(yr(-), yr(?))$ and $\mathcal{C}_\infty\mathcal{B}(m-, m?)$ are unital differential graded algebras, P is a unital differential graded algebra and the canonical projections above are morphisms of unital differential graded algebras. We have thus constructed a sequence of quasi-isomorphisms of unital differential graded algebras in $\mathsf{C}(\mathbb{B}, \mathbb{B})$

$$\mathcal{A}' \to \mathcal{C}_\infty\mathcal{B}(yr(-), yr(?)) \leftarrow P \to \mathcal{C}_\infty\mathcal{B}(m-, m?) \leftarrow \mathcal{C}_\infty\mathcal{B}(y-, y?) \leftarrow \mathcal{B}.$$

The quasi-isomorphisms of algebras being invertible up to homotopy in the category Alg_∞, we obtain a homologically unital A_∞-quasi-isomorphism

$$g' : \mathcal{A}' \to \mathcal{B}.$$

According to proposition 3.2.4.3, there exists a strictly unital A_∞-morphism g homotopic to g'. In particular, g is an A_∞-quasi-isomorphism. This is an A_∞-equivalence (see the first case.)

The general case: Let $\mathcal{A}$ and $\mathcal{B}$ be two strictly unital A_∞-categories over $\mathbb{A}$ and $\mathbb{B}$ and f an A_∞-functor such that f_1 is a quasi-isomorphism and induces an equivalence $H^0\mathcal{A} \to H^0\mathcal{B}$. Let us choose in each A_∞-isomorphism class $[A]$ of $\mathcal{A}$ a representative A_0 and denote by B_0 its image under $\dot{f}$. As $H^0 f : H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence, we deduce from the lemma (9.1.0.4) that any A_∞-isomorphism class $[B]$ in $\mathcal{B}$ admits a unique representative among the B_0. Let $\mathcal{A}'$ (resp. $\mathcal{B}'$) be the full subcategory of $\mathcal{A}$ (resp. $\mathcal{B}$) formed by A_0 (resp. B_0). The inclusion

$$\mathcal{A}' \to \mathcal{A} \quad \Big(\text{resp.} \quad \mathcal{B}' \to \mathcal{B}\Big)$$

is an A_∞-equivalence (see the second case). To show that f is an A_∞-equivalence, it suffices to show that the functor

$$f' : \mathcal{A}' \to \mathcal{B}'$$

induced by the A_∞-function f is an A_∞-equivalence. Its underlying map $\dot{f}'$ is a bijection and f'_1 is a quasi-isomorphism. We are therefore in the first case and f' is an A_∞-equivalence. $\square$

Chapter A

Model categories

In this appendix, we recall the definition, due to D. Quillen [Qui67], of a category of models (closed), some fundamental notions (fibrant objects, cofibrants, homotopies, Quillen functors) and some statements- keys. We then recall the examples we need in this manuscript. We refer to the book by M. Hovey [Hov99] and to the article by W. Dwyer and J. Spalinski [DS95] for more details.

Definitions and propositions

Definition A.6. Let E be a category. A *lifting (of g relative to f)* in the diagram

$$(1)\qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a morphism $\alpha : C \to B$ such that the two triangles in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dotted, "\alpha"] & D \end{tikzcd}$$

are commutative Let i and p be two morphisms in E. We will say that p *has the right-lifting property with respect to* i and that i *has the left-lifting property with respect to* p if any diagram of the form (1) admits a lift α.

Let $f : X \to X'$ and $g : Y \to Y'$ be two morphisms. The morphism f is a *retract* of g if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ X' \arrow[r] & Y' \arrow[r] & X' \end{tikzcd}$$

such that the horizontal compositions are the identity of X and the identity of X'.

Definition A.7. A *model category* is a quadruplet

$$(\mathsf{E}, \mathcal{E}q, \mathcal{F}ib, \mathcal{C}of),$$

where

- E is a category,
- $\mathcal{E}q$ is a class of morphisms called *weak equivalences*,
- $\mathcal{F}ib$ is a class of morphisms called *fibrations* (they are represented by double-headed arrows $\twoheadrightarrow$),
- $\mathcal{C}of$ is a class of morphisms called *cofibrations* (they are represented by arrows with a tail $\rightarrowtail$),

such that the axioms (CM1) – (CM5) below hold. A morphism belonging to $\mathcal{E}q \cap \mathcal{C}of$ will be called a *trivial cofibration* and a morphism from $\mathcal{E}q \cap \mathcal{F}ib$ will be called a *trivial fibration* .

(CM1) The category E admits all finite limits and all finite colimits.

(CM2) The class of weak equivalences is *saturated* , i.e. if two morphisms among f, g, fg are weak equivalences, the third one is also.

(CM3) The three classes of morphisms are stable under retracts.

(CM4) *lifting :*
a. Cofibrations have the left-lifting property with respect to the trivial fibrations,
b. Fibrations have the right-lifting property with respect to trivial cofibrations.

(CM5) *factorisation :*
a. Any morphism $f : A \to B$ factors into $f = pi$ where $i : A \rightarrowtail A'$ is a trivial cofibration and $p : A' \twoheadrightarrow B$ is a fibration.
b. Any morphism $f : A \to B$ factors into $f = pi$ where $i : A \rightarrowtail B'$ is a cofibration and $p : B' \twoheadrightarrow B$ is a trivial fibration.

Remark A.8. *We follow the terminology of [DS95] by calling "model category" what Quillen [Qui67], [Qui69] calls "closed model category". Note that the axioms are self-dual.*

Let $(\mathsf{E}, \mathcal{E}q, \mathcal{F}ib, \mathcal{C}of)$ a model category. We have the following properties:

- The category E has an initial object $\emptyset$ and a final object $*$.
- The fibrations are exactly the morphisms having the right lifting property with respect to the trivial cofibrations.
- The trivial fibrations are exactly the morphisms having the right lifting property with respect to the cofibrations.
- Cofibrations have dual lifting properties.

Definition A.9. *Let X be an object of E. A* cylinder *for X is an object $X \wedge I$ endowed with morphisms $i : X \coprod X \to X \wedge I$ and $p : X \wedge I\ raX$ such that*

1. *the morphism p is a weak equivalence,*
2. *the composition $p \circ i : X \coprod X \to X \wedge I \to X$ is the morphism*

$$[\mathbf{1}, \mathbf{1}] : X \coprod X \to X.$$

*Let $X \wedge I$ be a cylinder for X. Two morphisms $f, g : X \to Y$ of E are $X \wedge I$-*left-homotopes *if the morphism $[f, g] : X \coprod X \to Y$ factors into*

$$X \coprod X \xrightarrow{i} X \wedge I \xrightarrow{H} Y$$

*for a morphism H. Such a morphism H is called a $X \wedge I$-*left homotopy *from f to g. The morphisms f and g are* left homotopic *if they are $X \wedge I$-homotopic for a cylinder $X \wedge I$ for X. We will then write*

$$f \sim_l g.$$

The definition of a path object *for X is dual to that of a cylinder for X. The notion of* right homotopy *(denoted $\sim_r$) is dual to that of left homotopy.*

Definition A.10. An object X of E is *cofibrant* if the morphism $\emptyset \to X$ is a cofibration. It is *fibrant* if the morphism $X \to *$ is a fibration. The full subcategory of fibrant objects is denoted E_{f}, that of cofibrant objects E_{c} and that of fibrant and cofibrant objects is denoted E_{cf}.

Definition A.11. Let X be an object of E. A *cofibrant resolution of X* is a trivial fibration $X_{\mathsf{c}} \twoheadrightarrow X$, where X_{c} is cofibrant. A *fibrant resolution of X* is a trivial cofibration $Y \rightarrowtail X_{\mathsf{f}}$, where X_{f} is fibrant.

It follows from the axiom (CM5) that any object admits a cofibrant resolution and a fibrant resolution.

Lemma A.12. Let X be a cofibrant object and Y a fibrant object.

a. The relation of $X \wedge I$-left homotopy does not depend on the choice of cylinder $X \wedge I$. Similarly, the PY-right homotopy relation does not depend on the choice of path object PY.

b. The left homotopy and right homotopy relations coincide on $\mathsf{E}(X, Y)$. We define the *homotopy relation* $\sim$ as equal to these two relations.

c. The homotopy relation is an equivalence relation on $\mathsf{E}(X, Y)$.

d. Let X' be a cofibrant object and Y' a fibrant object. The relation $f \sim g$ implies $fh \sim gh$ and $h'f \sim h'g$ whatever the morphisms

$$h : X' \to X \quad \text{and} \quad h' : Y \to Y'.$$

□

The quotient $\mathsf{E}_{\mathsf{cf}}/\sim$ is therefore a category. We define the *homotopy category* $\mathsf{Ho}\,\mathsf{E}$ as the localization $\mathsf{E}[\mathcal{E}q^{-1}]$ of E with respect to the class of weak equivalences (see [GZ67, I.1]).

Proposition A.13. a. The inclusion $\mathsf{E}_{\mathsf{cf}} \to \mathsf{E}$ induces an equivalence

$$\mathsf{E}_{\mathsf{cf}}/\sim \longrightarrow \mathsf{Ho}\,\mathsf{E}.$$

b. Let X and Y be two objects of E. Let $X_{\mathsf{c}} \twoheadrightarrow X$ be a cofibrant resolution of X and $Y \rightarrowtail Y_{\mathsf{f}}$ a fibrant resolution of Y. We have a canonical bijection

$$\mathsf{Ho}\,\mathsf{E}\,(X,Y) \simeq \mathsf{E}\,(X_{\mathsf{c}}, Y_{\mathsf{f}})/\sim .$$

□

Quillen equivalence

Definition A.14. Let E and F be two model categories. A functor $G : \mathsf{E} \to \mathsf{F}$ is a *left Quillen functor* if it admits a right adjoint and if it preserves cofibrations and trivial cofibrations. A functor $D : \mathsf{F} \to \mathsf{E}$ is a *right Quillen functor* if it admits a left adjoint and if it preserves fibrations and trivial fibrations. Consider a pair of adjoint functors (G, D, ϕ), i.e. G is left adjoint to D and ϕ is a functorial bijection

$$\mathsf{Hom}_{\mathsf{F}}(GX, Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{E}}(X, DY).$$

It is called a *Quillen adjunction* if G is a left Quillen functor. (This implies that D is a right Quillen functor.) A Quillen adjunction is a *Quillen equivalence* if, for any cofibrant object X of E and any fibrant object Y of F, a morphism $f : GX \to Y$ is a weak equivalence if and only if $\phi f : X \to DY$ is a weak equivalence. We refer to [DS95, Sect. 9] for the details of the following definition.

Definition A.15. Let G be a left Quillen functor. The *left derived functor of* G is the functor

$$\mathbf{L}G : \mathsf{Ho}\,\mathsf{E} \longrightarrow \mathsf{Ho}\,\mathsf{F}$$

which sends an object X from E to GX_{c}, where $X_{\mathsf{c}} \twoheadrightarrow X$ is a cofibrant resolution of X. Let D be a right Quillen functor. The *right-derived functor of* D is the functor

$$\mathbf{R}D : \mathsf{Ho}\,\mathsf{F} \longrightarrow \mathsf{Ho}\,\mathsf{E}$$

which sends an object Y from F to GY_{f}, where $Y \rightarrowtail Y_{\mathsf{f}}$ is a fibrant resolution of Y.

Remark A.16. Note that if a functor G (resp. D) as in the definition preserves weak equivalences, then it induces a functor between the homotopic categories, and $\mathbf{L}G$ (resp. $\mathbf{R}D$) is canonically isomorphic to this induced functor.

Proposition A.17. Let (G, D, ϕ) be a Quillen adjunction from E to F. The following propositions are equivalent

a. (G, D, ϕ) is a Quillen equivalence.

b. The functors $\mathbf{L}G$ and $\mathbf{R}D$ are inverse equivalences of each other between $\mathsf{Ho}\,\mathsf{E}$ and $\mathsf{Ho}\,\mathsf{F}$.

Examples of model categories

Example A.18 (Complexes of C)**.** Let C be the base category (1.1.1). The category $\mathcal{C}\mathsf{C}$ of (1.1.1) admits a model category structure such that

- the class of weak equivalences is the class Qis of quasi-isomorphisms (note that these are exactly the morphisms which are invertible up to homotopy),

- the fibrations are the epimorphisms (i.e., the morphisms whose components are epimorphisms),

- the cofibrations are the monomorphisms (ie, the morphisms whose components are monomorphisms).

All the complexes are fibrant and cofibrant for this structure. The associated homotopy category is $\mathcal{H}\mathsf{C}$.

Example A.19 (Unbounded chain complexes)**.** Let R be a ring. Let $\mathcal{C}R$ be the category of chain complexes

$$\cdots \to M^{p-1} \to M^p \to M^{p+1} \to \cdots, \quad p \in \mathbf{Z},$$

of right R-modules. The following three classes of morphisms define a model category structure on $\mathcal{C}R$ (see [Hov99, Chap. 2]).

- Weak equivalences are quasi-isomorphisms.

- The fibrations are the morphisms $f : X \to Y$ such that f^n is surjective for all $n \in \mathbf{Z}$.

- The cofibrations are the morphisms which have the left-lifting property with respect to the trivial fibrations.

All the complexes are fibrant for this structure. If a complex X is cofibrant, then all its components X^n, $n \in \mathbf{Z}$, are projective. The converse is false. However, if we suppose that the complex X is bounded on the right and that its components are all projective, then it is cofibrant.

Chapter B

Obstruction theory

B.1 Obstruction Theory for A_∞-algebras

We study the obstruction theory of A_∞-algebras. Let C be a base category such as in chapter 1. $(A, m_1, \ldots, m_n)$ a A_n-algebra. It is a question of measuring the obstruction to the existence of a morphism $m_{n+1} : A^{\otimes n+1} \to A$ such that $(A, m_1, \ldots, m_{n+1})$ be a $\mathrm{A}_{\mathrm{n+1}}$-algebra (B.1.2). Let A and A' be two $\mathrm{A}_{\mathrm{n+1}}$-algebras. Consider a family of graded morphisms

$$f_i : A^{\otimes i} \to B, \qquad 1 \leq i \leq n,$$

defining an A_n-morphism $A \to A'$. We then measure the obstruction to the existence of a morphism $f_{n+1} : A^{\otimes n+1} \to A'$ such that the f_i, $1 \leq i \leq n+1$, define an $\mathrm{A}_{\mathrm{n+1}}$-morphism $A \to A'$ (B.1.5). We will show that this obstruction is functorial with respect to strict $\mathrm{A}_{\mathrm{n+1}}$-morphisms (B.1.6).

The study of obstructions is a classic tool, see for example T. Kadeishvili [Kad80], A. Prouté [Pro85]. It owes its existence to the fact that the A_∞-algebra operad is a minimal cofibrant model in the sense of M. Markl [Mar96] for the associative algebra operad. We do not adopt this point of view here, preferring a naïve approach.

A_∞-algebras

Let V be a graded object. Let there be graded morphisms

$$b_i : V^{\otimes i} \to V, \qquad 1 \leq i \leq n+1,$$

of degree $+1$. Let b denote the coderivation of $\overline{T^c_{[n+1]}}V$ given below

$$(b_1, \ldots, b_n, b_{n+1}).$$

Set

$$c(b_2, \ldots, b_n) = \sum_{2 \leq i \leq n} b_i(\mathbf{1}^{\otimes j} \otimes b_k \otimes \mathbf{1}^{\otimes l})$$

where the integers j, k, l satisfy $j+k+l = n+1$ and $j+1+l = i$. Recall that i_1 and p_{n+1} designate the canonical morphisms

$$V \longrightarrow \overline{T^c_{[n+1]}}V \qquad \text{and} \qquad \overline{T^c_{[n+1]}}V \longrightarrow V^{\otimes n+1}.$$

Lemma B.1.1. Suppose that the coderivation of the coalgebra $\overline{T^c_{[n]}}V$ given below

$$(b_1, \ldots, b_n)$$

is a differential.

a. The coderivation
$$b^2 : \overline{T^c_{[n+1]}}V \longrightarrow \overline{T^c_{[n+1]}}V$$
is equal to $i_1 \circ \zeta \circ p_{n+1}$, where $\zeta : V^{\otimes n+1} \to V$ is given by
$$\zeta = b_1 b_{n+1} + b_{n+1} b_1 + c(b_2, \ldots, b_n);$$
here the last occurrence of b_1 designates the differential of $(V, b_1)^{\otimes n+1}$.

b. The graded morphism $c(b_2, \ldots, b_n)$ is a cycle of
$$(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(V^{\otimes n+1}, V), \delta),$$
where the differential δ is induced by that of the complex (V, b_1).

In particular, the coderivation b is a differential if and only if the cycle $c(b_2, \ldots, b_n)$ is equal to the boundary $-\delta(b_{n+1})$.

Proof. a. Our hypothesis implies that the square b^2 is factorized by p_{n+1}. The image of the comultiplication Δ is included in

$$\overline{T^c_{[n]}}V \otimes \overline{T^c_{[n]}}V \subset \overline{T^c_{[n+1]}}V \otimes \overline{T^c_{[n+1]}}V.$$

We therefore have the equality

$$\Delta b^2 = (\mathbf{1} \otimes b^2 + b^2 \otimes \mathbf{1})\Delta = 0.$$

We deduce that the image of b^2 is included in $\ker \Delta = V$. This gives us the factorization by i_1. A direct calculation gives us the formula for ζ.

b. According to the first point, we have

$$b_1 \circ b^2 = b \circ b^2 = b^2 \circ b = b^2 \circ b_1,$$

where the last occurrence of b_1 denotes the differential of $(V, b_1)^{\otimes n+1}$. This shows that ζ is a cycle in the complex
$$(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(V^{\otimes n+1}, V), \delta).$$
As we have
$$\zeta = \delta(b_{n+1}) + c(b_2, \ldots, b_n)$$
the same is true for $c(b_2, \ldots, b_n)$. □

Corollary B.1.2. Let (A, m_1) be a complex. Consider graded morphisms

$$m_i : A^{\otimes i} \to A, \qquad 2 \leq i \leq n+1$$

of degree $2-i$. Suppose that the morphisms m_i, $1 \leq i \leq n$, define a structure of A$_\mathrm{n}$-algebra on A. The sub-expression
$$\sum_{i,k\neq 1} (-1)^{jk+l} m_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l})$$
of the equation $(*_{n+1})$ of (1.2.1.1) defines a cycle of $(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes n+1}, A), \delta)$. We denote it $r(m_2, \dots, m_n)$. The equation $(*_{n+1})$ is then rewritten
$$r(m_2, \dots, m_n) + \delta(m_{n+1}) = 0.$$

Proof. We apply the previous lemma to the graded space $V = SA$ and to the graded morphisms b_i defined using the bijections $b_i \leftrightarrow m_i$. These same bijections map the morphism $r(m_2, \dots, m_n)$ to the morphism $c(b_2, \dots, b_n)$ and the morphism $\delta(m_{n+1})$ to $\delta(b_{n+1})$. □

A$_\infty$-morphisms of A$_\infty$-algebras

The following lemmas are shown in a similar way.

Let V and W be two graded objects. Let b and b' be differentials of coalgebras on the coalgebras $\overline{T^c_{[n+1]}}V$ and $\overline{T^c_{[n+1]}}W$. Consider a family of graded morphisms
$$F_i : V^{\otimes i} \to W, \qquad 1 \leq i \leq n+1,$$
of degree 0. Let F be a morphism of coalgebras
$$\overline{T^c_{[n+1]}}V \longrightarrow \overline{T^c_{[n+1]}}W$$
that lifts the F_i. Put
$$c(F_1, \dots, F_n) = \sum_{k\geq 2} F_i(\mathbf{1}^{\otimes j} \otimes b_k \otimes \mathbf{1}^{\otimes l}) - \sum_{r\geq 2} b'_r(F_{i_1} \otimes \dots \otimes F_{i_r}),$$
where the integers j, k, l of the left sum satisfy $j+k+l = n+1$ and $j+1+l = i$, and where the sum of the integers i_r of the sum on the right is $n+1$.

Lemma B.1.3. Suppose that the morphism
$$F_{[n]} : \overline{T^c_{[n]}}V \to \overline{T^c_{[n]}}W$$
induced by F in n-primitives is compatible with differentials.

a. The (F,F)-coderivation
$$b'F - Fb : \overline{T^c_{[n+1]}}V \longrightarrow \overline{T^c_{[n+1]}}W$$
is equal to $i_1 \circ \zeta \circ p_{n+1}$, where $\zeta : V^{\otimes n+1} \to W$ is given by
$$\zeta = b_1 F_{n+1} + F_{n+1} b_1 + c(F_1, \dots, F_n);$$
here the last occurrence of b_1 designates the differential of $(V, b_1)^{\otimes n+1}$.

b. The graded morphism $c(F_1, \ldots, F_n)$ is a cycle of

$$(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(V^{\otimes n+1}, W), \delta),$$

where the differential δ is induced by those of the complexes (V, b_1) and (W, b_1').

In particular, the morphism F is compatible with coalgebra differentials if and only if we have

$$\delta(F_{n+1}) + c(F_1, \ldots, F_n) = 0.$$

□

Let's now look at the behavior of the obstruction with respect to the composition of the $\mathrm{A_{n+1}}$-morphisms.

Let V' and W' be two graded objects. Let d and d' be two differentials of coalgebras on the coalgebras $\overline{T^c_{[n+1]}}V'$ and $\overline{T^c_{[n+1]}}W'$. Consider two morphisms of differential graded coalgebras

$$G : \overline{T^c_{[n+1]}}V' \longrightarrow \overline{T^c_{[n+1]}}V \qquad \text{and} \qquad H : \overline{T^c_{[n+1]}}W \longrightarrow \overline{T^c_{[n+1]}}W'.$$

Direct calculations give us the following lemma.

Lemma B.1.4. a. We have equality

$$c(F_1, \ldots, F_n) \circ {G_1}^{\otimes n+1} + F_1 \circ \delta(G_{n+1}) = c((FG)_1, \ldots, (FG)_n)$$

of morphisms from $(V')^{\otimes n+1}$ into W.

b. We have equality

$$\delta(H_{n+1}) \circ {F_1}^{\otimes n+1} + H_1 \circ c(F_1, \ldots, F_n) = c((HF)_1, \ldots, (HF)_n)$$

of morphisms from $V^{\otimes n+1}$ into W'. □

Corollary B.1.5. Let A and B be two $\mathrm{A_{n+1}}$-algebras. Consider graded morphisms

$$f_i : A^{\otimes i} \to B, \qquad 1 \leq i \leq n+1,$$

of degree $1-i$. Suppose that the morphisms f_i, $1 \leq i \leq n$, define an A_n-morphism $A \to B$. The sub-expression

$$\sum_{k \neq 1} (-1)^{jk+l} f_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) - \sum_{r \neq 1} (-1)^s m_r(f_{i_1} \otimes \ldots \otimes f_{i_r})$$

of the equation $(**_{n+1})$ of (1.2.1.2) defines a cycle in $(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes n+1}, B), \delta)$. We denote it $r(f_1, \ldots, f_n)$. The equation $(**_{n+1})$ can be rewritten

$$r(f_1, \ldots, f_n) + \delta(f_{n+1}) = 0.$$

□

Corollary B.1.6. Let A' and B' be two $\mathrm{A_{n+1}}$-algebras. Let $g : A' \to A$ and $h : B \to B'$ be two strict $\mathrm{A_{n+1}}$-morphisms. We have the equalities of morphisms

1. $r(f_1, \ldots, f_n) \circ {g_1}^{\otimes n+1} = r((fg)_1, \ldots, (fg)_n)$,
2. $h_1 \circ r(f_1, \ldots, f_n) = r((hf)_1, \ldots, (hf)_n)$.

Obstruction is therefore functorial with respect to strict morphisms.

Proof. This is the translation of the lemma B.1.6 applied to the bar constructions of the algebras A, A', B and B'. The morphisms g and h being strict, we have $H_{n+1} = 0$ and $G_{n+1} = 0$. The equations of (B.1.6) are then translated by those of the corollary. $\square$

B.2 Obstruction theory for polydules

The proofs of this section being almost identical to those of the section 1.2.2, we content ourselves with stating the results. Let C and C' be the basic categories of the section 2.1.

Lemma B.2.1. Let A be a $\mathrm{A_n}$-algebra. Let (M, m_1^M) be a complex. Consider graded morphisms

$$m_i^M : M \otimes A^{\otimes i-1} \to M, \quad 2 \leq i \leq n+1,$$

of degree $2-i$. Suppose that the morphisms m_i, $1 \leq i \leq n$, define a structure of $\mathrm{A_n}$-module on M. The sub-expression

$$\sum_{i,k \neq 1} (-1)^{jk+l} m_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l})$$

of the equation $({*'_{n+1}})$ of (2.3.1.2) defines a cycle of $(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(M \otimes A^{\otimes n}, M), \delta)$, where δ is induced by m_1^A and m_1^M. We denote it $r(m_2, \ldots, m_n)$. The equation $({*'_{n+1}})$ is then rewritten

$$r(m_2, \ldots, m_n) + \delta(m_{n+1}) = 0.$$

$\square$

Lemma B.2.2. Let A be a $\mathrm{A_n}$-algebra. Let M and N be two $\mathrm{A_{n+1}}$-modules on A. Consider graded morphisms

$$f_i : M \otimes A^{\otimes i-1} \to N, \qquad 1 \leq i \leq n+1,$$

of degree $1-i$. Suppose that the morphisms f_i, $1 \leq i \leq n$, define an A_n-morphism $M \to N$. The sub-expression

$$\sum_{k \neq 1} (-1)^{jk+l} f_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) = \sum_{s \neq 0} m_{s+1}(f_r \otimes \mathbf{1}^{\otimes s})$$

of the equation $({**'_{n+1}})$ of (2.3.1.5) defines a cycle in

$$(\mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(M \otimes A^{\otimes n}, N), \delta).$$

We denote it $r(f_1, \ldots, f_n)$. The equation $({**'_{n+1}})$ is then rewritten

$$r(f_1, \ldots, f_n) + \delta(f_{n+1}) = 0.$$

$\square$

Lemma B.2.3. Let A be a $\mathrm{A_n}$-algebra. Let M' and N' be two $\mathrm{A_{n+1}}$-modules. Let $g : M' \to M$ and $h : N \to N'$ be two strict $\mathrm{A_{n+1}}$-morphisms. We have the equalities of morphisms

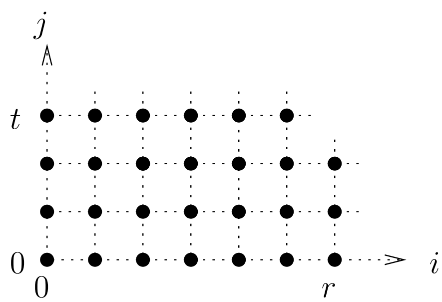
1. $r(f_1, \ldots, f_n) \circ g_1 \otimes \mathbf{1}^{\otimes n} = r((fg)_1, \ldots, (fg)_n)$,
2. $h_1 \circ r(f_1, \ldots, f_n) = r((hf)_1, \ldots, (hf)_n)$.

□

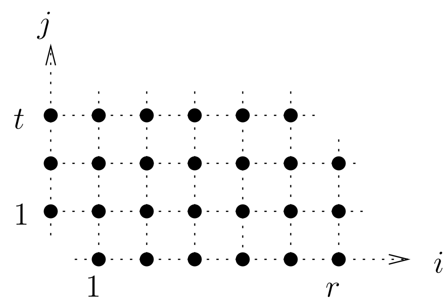
B.3 Obstruction theory for bipolydules

The proofs in this section are omitted because they are similar to those in the B.1 section. Let C, C' and C'' be the basic categories of the section 2.5.

Let A and A'' be two A_∞-algebras in C and C''. In what follows, r and t denote two integers ≥ 0 and $\mathcal{E}$ denotes the set of pairs of integers (i, j) such that $0 \leq i \leq r$ and $0 \leq j \leq t-1$, or, $0 \leq i \leq r-1$ and $0 \leq j \leq t$ (see graphic below). The set $\mathcal{E}'$ is equal to $\mathcal{E} \setminus (0, 0)$.



set $\mathcal{E}$



set $\mathcal{E}'$

Let M be a graded differential object of C'. We note its differential $m_{0,0}$. Consider

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A''^{\otimes j} \to M \to M, \quad 0 \leq j \leq t, \quad 0 \leq i \leq r, \quad (i,j) \neq (0,0),$$

a graded morphism of degree $1 - i - j$ in C'.

Lemma B.3.1. Suppose that the morphisms $m_{i,j}$, $(i, j) \in \mathcal{E}'$, satisfy the equations $(*''_{k,l})$, $(k, l)\ in \mathcal{E}$, of the definition 2.5.1.1. The sub-expression

$$\sum_{* \notin \{1,(0,0),(r,t)\}} (-1)^{j+i(|m_*|)} m_{\bullet,\bullet}(\mathbf{1}^{\otimes i} \otimes m_* \otimes \mathbf{1}^{\otimes j})$$

of the equation $(*''_{r,t})$ defines a cycle of

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(A^{\otimes r} \otimes M \otimes A''^{\otimes t}, M).$$

We denote it $c\big(m_{i,j},\ (i,j) \in \mathcal{E}'\big)$. The morphisms $m_{i,j}$, $0 \leq j \leq t$, $0 \leq i \leq r$, satisfy the equation $(*''_{r,t})$ if and only if we have equality

$$\delta(m_{r,t}) = c\big(m_{i,j},\ (i,j) \in \mathcal{E}'\big).$$

□

Let M and N be two A-A''-bipolydules in C'. Consider

$$f_{i,j} : A^{\otimes i} \otimes M \otimes A''^{\otimes j} \to M \to M, \quad 0 \leq j \leq t, \quad 0 \leq i \leq r,$$

a graded morphism of degree $-i-j$ in $\mathcal{G}r\mathsf{C}'$.

Lemma B.3.2. Suppose that the morphisms $f_{i,j}$, $(i,j) \in \mathcal{E}$, satisfy the equations $(**''_{k,l})$, $(k,l)\ in \mathcal{E}$, of the definition 2.5.1.1. The sub-expression

$$\begin{aligned}\textstyle\sum_{(\alpha,\beta)\neq(0,0)}(-1)^{\alpha(-i-j)} m_{\alpha,\beta}(\mathbf{1}^{\otimes\alpha} \otimes f_{k,l} \otimes \mathbf{1}^{\otimes\beta}) = \\ \textstyle\sum_{*\notin\{1,(0,0)\}}(-1)^{j+i(|m_*|)} f_{\bullet,\bullet}(\mathbf{1}^{\otimes i} \otimes m_* \otimes \mathbf{1}^{\otimes j})\end{aligned}$$

of the equation $(**''_{r,t})$ is a cycle of

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}'}(A^{\otimes r} \otimes M \otimes A''^{\otimes t}, N).$$

We denote it $c'\big(f_{i,j},\ (i,j) \in \mathcal{E}\big)$. The morphisms $f_{i,j}$, $0 \leq j \leq t$, $0 \leq i \leq r$ satisfy the equation $(**''_{r,t})$ if and only if we have equality

$$\delta(f_{r,t}) = c'\big(f_{i,j},\ (i,j) \in \mathcal{E}\big).$$

□

We now look at the compatibility of the obstruction with strict morphisms.

Let M' and N' be two A-A'-bipolydules and

$$g : M' \to M \quad \text{and} \quad h : N \to N'$$

be two strict A_∞-morphisms of bipolydules given by graded morphisms of degree 0 in $\mathcal{G}r\mathsf{C}'$

$$g_{0,0} : M' \to M \quad \text{and} \quad h_{0,0} : N \to N'.$$

The morphisms

$$(f \circ g)_{i,j} \quad \text{and} \quad (h \circ f)_{i,j}, \quad 0 \leq j \leq t, \quad 0 \leq i \leq r,$$

are defined by the same formulas as those giving the composition of the morphisms of bipolydules.

Lemma B.3.3. We have the following equalities:

1. $c'\big(f_{i,j},\ (i,j) \in \mathcal{E}\big) \circ (\mathbf{1}^{\otimes r} \otimes g_{0,0} \otimes \mathbf{1}^{\otimes t}) = c'\big((f \circ g)_{i,j},\ (i,j) \in \mathcal{E}\big)$,
2. $h_{0,0} \circ c'\big(f_{i,j},\ (i,j) \in \mathcal{E}\big) = c'\big((h \circ f)_{i,j},\ (i,j) \in \mathcal{E}\big)$.

□

B.4 Hochschild cohomology and obstruction theory for minimal A_∞-structures

In this section, we recall the Hochschild cohomology of a graded algebra with coefficients in a graded bimodule. We then establish an obstruction theory of minimal A_∞-algebras (resp. of A_∞-morphisms between minimal A_∞-algebras and of homotopies between these A_∞-morphisms).

Remainder on Hochschild cohomology

Let C be a base category such as in chapter 1. Let $A \in \mathcal{G}r\mathsf{C}$ be an associative algebra. Consider A as an A_∞-algebra with $m_2 = \mu^A$ and $m_i = 0$ for all $i \neq 2$. Recall that $(BA)^+$ is the co-augmented bar construction of A. Let $\mathsf{coder}((BA)^+)$ be the space of coderivations $(BA)^+ \to (BA)^+$. It is graded by the degree of co-derivations. The map

$$\delta : D \mapsto b^A \circ D - (-1)^{|D|} D \circ b^A,$$

where b^A is the differential of $(BA)^+$ and D is of degree $|D|$, defines a differential on $\mathsf{coder}((BA)^+)$. We show (as in the lemma 1.1.2.2) that we have a natural bijection

$$\begin{array}{ccc} \mathsf{coder}((BA)^+) & \xrightarrow{\sim} & \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((BA)^+, SA) \\ D & \mapsto & p_1 \circ D. \end{array}$$

Thus, a coderivation D is determined by the components of $p_1 \circ D$

$$D_i : (SA)^{\otimes i} \to SA, \quad i \geq 0.$$

The bijections $b_i \leftrightarrow m_i$, $i \geq 1$, of the section 1.2.2 (completed with the bijection which associates to the morphism $b_0 : e \to SA$ the morphism $m_0 = -\omega b_0 : e \to A$) give us a bijection

$$\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((BA)^+, SA) \xrightarrow{\sim} \prod_{i \geq 0} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes i}, A).$$

The *Hochschild complex* is defined by these bijections as

$$C(A, A) = S^{-1} \prod_{i \geq 0} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes i}, A).$$

Its differential δ_{Hoch} sends a morphism $f : A^{\otimes n} \to A$ of degree r to the morphism

$$\delta_{Hoch}(f) : A^{\otimes n+1} \longrightarrow A$$

given by the sum

$$\sum (-1)^{r+n+k} f_i(\mathbf{1}^{\otimes j} \otimes \mu \otimes \mathbf{1}^{\otimes k}) + (-1)^{r+n+1} \mu(\mathbf{1} \otimes f_i) + (-1)^r \mu(f_i \otimes \mathbf{1}).$$

If the degree of f is zero, we find the usual definition (see for example [CE99, Chap. IX]). Let $M \in \mathcal{G}r\mathsf{C}$ be an A-A-bimodule. The *Hochschild complex with coefficients in* M is the space

$$C(A, M) = \prod_{i \geq 0} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(A^{\otimes i}, M),$$

its grading is induced by the grading of the space

$$\prod_{i \geq 0} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((SA)^{\otimes i}, SM)$$

and its differential δ_{Hoch} is defined by the same formula as before. The *Hochschild cohomology of* A *with coefficients in* M is the cohomology of $C(A, M)$. If A is unital, the complex $C(A, M)$ is homotopically equivalent to the *reduced Hochschild subcomplex* (see [CE99, Chap. IX])

$$\overline{C}(A, M) = \prod_{i \geq 0} \mathsf{Hom}_{\mathcal{G}r\mathsf{C}}(\overline{A}^{\otimes i}, M),$$

where $\overline{A}$ is the unit of the cokernel of A. The differential of $\overline{C}(A, M)$ is induced by that of $C(A, M)$.

Obstruction to the extension of a $\mathrm{A_n}$-minimal algebra into a $\mathrm{A_{n+1}}$-minimal algebra

Lemma B.4.1. Let A be a graded algebra of $\mathcal{G}r\mathsf{C}$. Consider graded morphisms

$$m_i : A^{\otimes i} \to A, \quad 3 \leq i \leq n,$$

of degree $2-i$. We set $m_2 = \mu^A$. Suppose that the morphisms m_i, $2 \leq i \leq n-1$, define a structure of $\mathrm{A_n}$-minimal algebra on A. The sub-expression

$$\sum_{i,k \notin \{1,2\}} (-1)^{j+kl} m_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l})$$

of the equation $(*_{n+1})$ of (1.2.1.1) defines a Hochschild cycle. We denote it $r(m_3, \ldots, m_{n-1})$. The equation $(*_{n+1})$ is then rewritten

$$\delta_{Hoch}(m_n) + r(m_3, \ldots, m_{n-1}) = 0.$$

Proof. Consider the sequence of morphisms b_i, $2 \leq i \leq n$, given by the bijections $b_i \leftrightarrow m_i$ (see 1.2.2). We denote by D the coderivation of $(BA)^+$ such that the components of $p_1 \circ D$ are given by the sequence

$$(0, 0, b_2, \ldots, b_{n-1}, b_n, 0, \ldots).$$

As the m_i, $2 \leq i \leq n-1$, define a minimal $\mathrm{A_n}$-algebra structure, the square of the coderivation D restricted to the sub-cogebra $\overline{T^c_{[n]}}SA$ is zero. We deduce that the composition

$$\Delta \circ D^2 = (\mathbf{1} \otimes D^2 + D^2 \otimes \mathbf{1}) \circ \Delta$$

vanishes on the subspace $(SA)^{\otimes n+1}$. It follows that the image by D^2 of the subspace $(SA)^{\otimes n+1}$ is contained in $\ker \Delta = SA$ and that of the subspace $(SA)^{\otimes n+2}$ is contained in $(SA)^{\otimes 2} \oplus SA$. Thus, on the subspace $(SA)^{\otimes n+2}$, we have the equality

$$D^2 \circ b_2 = D^3 = b_2 \circ D^2.$$

This shows that the element

$$D^2|_{(SA)^{\otimes n+1}} \in \mathsf{Hom}((SA)^{\otimes n+1}, SA)$$

is a cycle. The first assertion of the lemma is deduced from the fact that the element

$$\omega(D^2|_{(SA)^{\otimes n+1}})$$

corresponds to the element $r(m_3, \ldots, m_{n-1})$ by the isomorphism of complexes

$$S^{-1}\mathsf{Hom}_{\mathcal{G}r\mathsf{C}}((BA)^+, SA) \xrightarrow{\sim} C(A, A).$$

The last assertion of the lemma is immediate. $\square$

Obstruction to the extension of an $\mathrm{A_n}$-morphism between A_∞-minimal algebras into an $\mathrm{A_{n+1}}$-morphism

Lemma B.4.2. Let A and A' be two minimal A_∞-algebras. Let

$$g : (A, m_2) \to (A', m_2')$$

a morphism of graded algebras. Consider graded morphisms

$$f_i : A^{\otimes i} \to A', \quad 2 \leq i \leq n,$$

of degree $1-i$. We set $f_1 = g$. Suppose that the morphisms f_i, $1 \leq i \leq n-1$, define an $\mathrm{A_n}$-morphism $A \to A'$. The sub-expression

$$\sum_{k \notin \{1,2\}} (-1)^{jk+l} f_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) - \sum_{r \notin \{1,2\}} (-1)^s m'_r(f_{i_1} \otimes \ldots \otimes f_{i_r})$$

of the equation $(**_{n+1})$ of (1.2.1.2) defines a Hochschild cycle in $C(A, A')$; the structure of A-bimodule on A' is given by g. We denote this cycle $r(f_2, \ldots, f_{n-1})$. The equation $(**_{n+1})$ is then rewritten

$$\delta_{Hoch}(f_n) + r(f_2, \ldots, f_{n-1}) = 0.$$

□

Obstruction to the extension of an $\mathrm{A_n}$-homotopy between A_∞-morphisms of A_∞-minimal algebras into an $\mathrm{A_{n+1}}$-homotopy

Lemma B.4.3. Let A and A' be two minimal A_∞-algebras. Let f and g be two A_∞-morphisms $A \to A'$. Let there be graded morphisms

$$h_i : A^{\otimes i} \to A', \quad 2 \leq i \leq n,$$

of degree $-i$. Set $h_1 = 0$. Suppose that the morphisms h_i, $1 \leq i \leq n-1$, define a homotopy between f and g considered as $\mathrm{A_n}$-morphism $A \to A'$ (we then have $f_1 = g_1$). The sub-expression

$$\Bigg(- \sum_{r+1+t \notin \{1,2\}} (-1)^s m_{r+1+t}(f_{i_1} \otimes \ldots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \ldots \otimes g_{i_t}) + \\ - \sum_{k \notin \{1,2\}} (-1)^{jk+l} h_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}) + f_{n+1} - g_{n+1} \Bigg)$$

of the equation $(***_{n+1})$ of the definition 1.2.1.7 defines a Hochschild cycle in $C(A, A')$; the A-bimodule structure on A' is given by f_1 and g_1. We denote this cycle $r(h_2, \ldots, h_{n-1})$. The equation $(***_{n+1})$ is then rewritten

$$\delta_{Hoch}(h_n) + r(h_2, \ldots, h_{n-1}) = 0.$$

□

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Notations

Les notations de base

$\mathbb{K}$	corps de base	20
C, C', $\mathsf{C}(\mathbb{O},\mathbb{O})$	catégories monoïdales ambiantes	20, 62, 112
$\otimes$, $\otimes_{\mathbb{O}}$	produit tensoriel	20, 112
e, $e_{\mathbb{O}}$	élément neutre pour le produit tensoriel	20, 112
$\mathcal{G}r\mathsf{C}$	catégorie des objets gradués de C	20
$\mathcal{C}\mathsf{C}$	catégorie des objets différentiels gradués de C	21
C	bicatégorie ambiante (à partir du chapitre 4)	112, 133
S	suspension des objets de $\mathcal{G}r\mathsf{C}$ et $\mathcal{C}\mathsf{C}$	21
s	morphisme de foncteurs $\mathbf{1} \to S$	21
$\omega = s^{-1}$	morphismes de foncteur $S \to \mathbf{1}$	21
$C(f)$	cône d'un morphisme f	76
$\delta(f)$	bord d'un morphisme gradué entre deux complexes	21

Les catégories de modèles et catégories triangulées

$\mathcal{E}q$	classe des équivalences faibles	200
$\mathcal{C}of$	classe des cofibrations	200
$\mathcal{F}ib$	classe des fibrations	200
E_{c}, E_{f}, E_{cf}	sous-catégories des objets cofibrants, des objets fibrants et de objets cofibrants et fibrants de E	201
$\mathsf{Ho}\,\mathsf{E}$	catégorie homotopique de E	201
$\mathsf{tria}\,\mathbb{A}$	sous-catégorie triangulée engendrée par les objets de $\mathbb{A}$	171
$\mathsf{Tria}\,\mathbb{A}$	sous-catégorie triangulée (aux sommes infinies) engendrée par les objets de $\mathbb{A}$	171

Les A_∞-algèbres et les algèbres

η	unité	62
ε	morphisme d'augmentation	62
A^+	augmentation de A	62
$\overline{A}$	réduction d'une algèbre augmentée	62
$\overline{T}V$, TV	algèbre tensorielle réduite et augmentée	22, 83
BA	construction bar réduite d'une A_∞-algèbre	28

B^+A	construction bar augmentée d'une A_∞-algèbre augmentée	83, 68
b^A, b	différentielle de la construction bar	28
$(*_m)$, $(**_m)$, $(***_m)$	équations de type A_∞	24, 25, 26
$r(m_\bullet, \ldots, m_n)$	cycle mesurant les obstructions	206, 213
$r(f_\bullet, \ldots, f_n)$	cycle mesurant les obstructions	208, 214
UA	algèbre enveloppante d'une A_∞-algèbre A	84
Alg	catégorie des algèbres différentielles graduées	23
Alga	catégorie des algèbres dg augmentées dont les morphismes sont augmentés	63
Alg_∞	catégorie des A_∞-algèbres	25
Alga_∞	catégorie formée des A_∞-algèbres augmentées dont les morphismes sont augmentés	81
$(\mathsf{Alg}_\infty)_{hu}$	catégorie des A_∞-algèbres homologiquement unitaires dont les morphismes sont homologiquement unitaires	106
$(\mathsf{Alg}_\infty)_{u}$	catégorie des A_∞-algèbres strictement unitaires dont les morphismes sont homologiquement unitaires	106
$(\mathsf{Alg}_\infty)_{su}$	catégorie des A_∞-algèbres strictement unitaires dont les morphismes sont strictement unitaires	106
$A \rightarrowtail A\langle M, \alpha\rangle$	cofibration standard de Alg	37
$C^*(A, M)$	complexe de Hochschild de A à coefficients dans M	212
$\overline{C}^*(A, M)$	complexe de Hochschild réduit	212
δ_{Hoch}	bord de Hochschild	212
τ	cochaîne tordante	67
τ_A, τ_C	cochaînes tordantes universelles	69

Les A_∞-cogèbres et les cogèbres

$C_{[n]}$	n-primitifs de C	23
η	co-unité	**??**
ε	morphisme de co-augmentation	64
C^+	co-augmentation d'une cogèbre C	64
$\overline{C}$	réduction d'une cogèbre co-augmentée C	64
$\overline{T^c}V$, T^cV	cogèbre tensorielle réduite et co-augmentée	23, 65
$\overline{T^c_{[n]}}V$	n-primitifs de la cogèbre $\overline{T^c}C$	29
ΩC, Ω^+C	construction cobar réduite et co-augmentée	30, 68, 68
Cog	catégorie des cogèbres différentielles graduées	24
Cogc	catégorie des cogèbres dg cocomplètes	24
Qis	classe des quasi-isomorphismes	51
$Qisf$	classe des quasi-isomorphismes filtrés	51
Cog_∞	catégorie des A_∞-cogèbres	30

Les polydules, les bipolydules et les modules

BM	construction bar d'un A-polydule	83
$R_\tau M$, RM	produit tensoriel tordu $M \otimes_\tau C$	67
$M \otimes_\tau C$	produit tensoriel tordu	67
$r(m_2, \ldots, m_n)$	cycle mesurant les obstructions	209
$r(f_1, \ldots, f_n)$	cycle mesurant les obstructions	209
$\overset{\infty}{\mathsf{Hom}}_{A'}(X, -)$	foncteur standard	113
$? \overset{\infty}{\otimes}_A X$	foncteur standard	114
$\mathsf{Mod}\, A$	catégorie des A-modules différentiels gradués unitaires	63
$\mathcal{D}A$	catégorie dérivée de $\mathsf{Mod}\, A$	76
$\mathsf{Nod}_\infty\, A$	catégorie des A-polydules (non nécessairement strictement unitaires)	80
$\left(\mathsf{Nod}_\infty\, A\right)_u$	sous-catégorie pleine de $\mathsf{Nod}_\infty\, A$ formée des A-polydules strictement unitaires	109
$\mathsf{Nod}_\infty^{\mathsf{strict}}\, A$	catégorie ayant les mêmes objets que $\mathsf{Nod}_\infty\, A$ et dont les morphismes sont les A_∞-morphismes stricts	80
$\mathsf{Mod}_\infty\, A$	catégorie des A-polydules strictement unitaires	81
$\mathsf{Mod}_\infty^{\mathsf{strict}}\, A$	$\mathsf{Nod}_\infty^{\mathsf{strict}}\, A \cap \mathsf{Mod}_\infty\, A$	81
$\mathcal{H}_\infty A$	catégorie $\mathsf{Mod}_\infty\, A$ quotientée par la relation d'homotopie	122
$\mathcal{D}_\infty A$	catégorie dérivée de $\mathsf{Mod}_\infty\, A$	118
$\mathcal{N}_\infty A$	catégorie différentielle graduée des A-polydules (non nécessairement strictement unitaires)	138
$\left(\mathcal{N}_\infty A\right)_u$	sous-catégorie de $\mathcal{N}_\infty A$ formée des A-polydules strictement unitaires	138
$\mathcal{C}_\infty A$	catégorie différentielle graduée des A-polydules strictement unitaires	138
$\mathsf{Mod}(A, A')$	catégorie des A-A'-bimodules dg unitaires	93
$\mathsf{Nod}_\infty(A, A')$	catégorie des A-A'-polydules (non nécessairement strictement unitaires)	92
$\left(\mathsf{Nod}_\infty(A, A')\right)_u$	sous-catégorie pleine de $\mathsf{Nod}_\infty(A, A')$ formée des A-A'-bipolydules strictement unitaires	110
$\mathsf{Mod}_\infty(A, A')$	catégorie des A-A'-polydules strictement unitaires	92
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$\left(\mathcal{N}_\infty(A, A')\right)_u$	sous-catégorie pleine de $\mathcal{N}_\infty(A, A')$ formée des A-A'-bipolydules strictement unitaires	139
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$N \otimes_\tau A$	produit tensoriel tordu	67
$\square_C$, $\square$	produit cotensoriel au-dessus de C	114
$\mathsf{Com}\, C$	catégorie des comodules dg unitaires	65
$\mathsf{Comc}\, C$	catégorie des comodules dg cocomplets sur C	66
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Sur les A_∞-catégories

Kenji Lefèvre-Hasegawa

Résumé : Nous étudions les A_∞-algèbres $\mathbf{Z}$-graduées (non nécessairement connexes) et leurs A_∞-modules. En utilisant les constructions bar et cobar ainsi que les outils de l'algèbre homotopique de Quillen, nous décrivons la localisation de la catégorie des A_∞-algèbres par rapport aux A_∞-quasi-isomorphismes. Nous adaptons ensuite ces méthodes pour décrire la catégorie dérivée $\mathcal{D}_\infty A$ d'une A_∞-algèbre augmentée A. Le cas où A n'est pas muni d'une augmentation est traité différemment. Néanmoins, lorsque A est strictement unitaire, sa catégorie dérivée peut être décrite de la même manière que dans le cas augmenté. Nous étudions ensuite deux variantes de la notion d'unitarité pour les A_∞-algèbres : l'unitarité stricte et l'unitarité homologique. Nous montrons que d'un point de vue homotopique, il n'y a pas de différence entre ces deux notions. Nous donnons ensuite un formalisme qui permet de définir les A_∞-catégories comme des A_∞-algèbres dans certaines catégories monoïdales. Nous généralisons à ce cadre les constructions fondamentales de la théorie des catégories : le foncteur de Yoneda, les catégories de foncteurs, les équivalences de catégories... Nous montrons que toute catégorie triangulée algébrique engendrée par un ensemble d'objets est A_∞-prétriangulée, c'est-à-dire qu'elle est équivalente à $H^0\mathsf{tw}\mathcal{A}$, où $\mathsf{tw}\mathcal{A}$ est l'A_∞-catégorie des objets tordus d'une certaine A_∞-catégorie $\mathcal{A}$.

Discipline : mathématiques

Mots-clés : A_∞-catégorie, algèbre à homotopie près, catégorie dérivée, algèbre homologique, catégorie triangulée, construction bar

On A_∞-categories

Kenji Lefèvre-Hasegawa

Abstract : We study (not necessarily connected) $\mathbf{Z}$-graded A_∞-algebras and their A_∞-modules. Using the cobar and the bar construction and Quillen's homotopical algebra, we describe the localisation of the category of A_∞-algebras with respect to A_∞-quasi-isomorphisms. We then adapt these methods to describe the derived category $\mathcal{D}_\infty A$ of an augmented A_∞-algebra A. The case where A is not endowed with an augmentation is treated differently. Nevertheless, when A is strictly unital, its derived category can be described in the same way as in the augmented case. Next, we compare two different notions of A_∞-unitarity : strict unitarity and homological unitarity. We show that, up to homotopy, there is no difference between these two notions. We then establish a formalism which allows us to view A_∞-categories as A_∞-algebras in suitable monoidal categories. We generalize the fundamental constructions of category theory to this setting : Yoneda embeddings, categories of functors, equivalences of categories... We show that any algebraic triangulated category $\mathcal{T}$ which admits a set of generators is A_∞-pretriangulated, that is to say, $\mathcal{T}$ is equivalent to $H^0\mathsf{tw}\mathcal{A}$, where $\mathsf{tw}\mathcal{A}$ is the A_∞-category of twisted objets of a certain A_∞-category $\mathcal{A}$.

Keywords: A_∞-categorie, homotopy algebra, derived category, homological algebra, triangulated category, bar construction

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