1. (20 points) Define $\mathbf{u} = \langle -1, 2, 5\rangle$ and $\mathbf{v} = \langle 3, 2, 1\rangle$. Compute the following.

(a) $\|\mathbf{u}\|$.

Solution: $\|\mathbf{u}\| = \sqrt{1+4+25} = \boxed{\sqrt{30}}$

(b) $\mathbf{u}\cdot\mathbf{v}$.

Solution: $\mathbf{u}\cdot\mathbf{v} = (-1)3 + 2(2) + 5(1) = \boxed{6}$

(c) $\mathbf{u}\times\mathbf{v}$.

Solution: $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 5 \\ 3 & 2 & 1 \end{vmatrix} = \left\langle 2-10, -(-1-15), -2-6\right\rangle = \boxed{\langle -8, 16, -8\rangle}$

(d) The area of the triangle formed by $\mathbf{u}$ and $\mathbf{v}$.

Solution: Area $= \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{1}{2}\sqrt{64+16^2+64} = \frac{1}{2}\sqrt{8^2(1+4+1)} = \boxed{4\sqrt{6}}$

(e) The angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{6}{\sqrt{30}\sqrt{14}} = \frac{6}{2\sqrt{15}\sqrt{7}} = \frac{3}{\sqrt{105}}$$

$$\implies \theta = \boxed{\cos^{-1}\left(\frac{3}{\sqrt{105}}\right)}$$

2. (20 points) Let $\mathbf{u} = \langle 1, 2 \rangle$, and $\mathbf{v}$ be a vector of length 2 which is at an angle of $\frac{\pi}{3}$ to $\mathbf{u}$. Compute the following:

(a) $\mathbf{u} \cdot \mathbf{v}$

Solution: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta = \sqrt{5}(2)\cos\frac{\pi}{3} = \boxed{\sqrt{5}}$

(b) $\|\mathbf{u} \times \mathbf{v}\|$

Solution: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta = \sqrt{5}(2)\sin\frac{\pi}{3} = \boxed{\sqrt{15}}$

(c) Assuming we let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, compute the following:

(i) $\mathbf{u} \cdot \mathbf{w}$

Solution: Since $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, $\mathbf{w} \perp \mathbf{u}$, so $\mathbf{u} \cdot \mathbf{w} = \boxed{0}$

(ii) $\|\mathbf{u} \times \mathbf{w}\|$

Solution: In part (b), we computed $\|\mathbf{w}\| = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{15}$.
Thus $\|\mathbf{u} \times \mathbf{w}\| = \|\mathbf{u}\|\|\mathbf{w}\| \sin\frac{\pi}{2} = \sqrt{5}\sqrt{15} = \boxed{5\sqrt{3}}$

3. (30 points)

(a) Find an equation for the plane containing the points $P = (1, 0, 2)$, $Q = (-1, 3, 3)$, and $R = (0, -1, 1)$. Express your answer in the form $Ax + By + Cz = D$.

Solution:

$$\overrightarrow{PQ} = \langle -2, 3, 1 \rangle$$
$$\overrightarrow{PR} = \langle -1, -1, -1 \rangle$$
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ -1 & -1 & -1 \end{vmatrix} = \langle -2, -3, 5 \rangle = \mathbf{n}$$

Using the point P,

$$\begin{aligned} & \langle -2, -3, 5 \rangle \cdot \langle x-1, y, z-2 \rangle = 0 \\ \implies \quad & -2(x-1) - 3y + 5(z-2) = 0 \\ \implies \quad & \boxed{-2x - 3y + 5z = 8} \end{aligned}$$

(Other choices of normal vector $\mathbf{n}$ may be used, which will multiply the boxed equation by a scalar.)

(b) Find the shortest distance from point $S = (2, -1, -1)$ to the plane you found in (a).

Solution: The distance is given by

$$\left\| \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} \right\| = \frac{|\overrightarrow{PS} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{\left| \langle 1, -1, -3 \rangle \cdot \langle -2, -3, 5 \rangle \right|}{\sqrt{4 + 9 + 25}} = \boxed{\frac{14}{\sqrt{38}}}$$

(c) Find the equation for the line passing through point R and perpendicular to the plane you found in (a).

Solution: $\mathbf{r}(t) = \langle 0, -1, 1 \rangle + t \langle -2, -3, 5 \rangle$, $t \in \mathbb{R}$

4. (20 points) Convert the equation written in spherical coordinates into an equation in Cartesian coordinates.

$$\tan(\varphi)(\cos(\theta) - 2\sin(\theta)) = \rho$$

Solution:

$$\begin{aligned} &\Longrightarrow & \sin\varphi(\cos\theta - 2\sin\theta) &= \rho\cos\varphi \\ &\Longrightarrow & \rho\sin\varphi\cos\theta - 2\rho\sin\theta\sin\varphi &= \rho^2\cos\varphi \\ &\Longrightarrow & &\boxed{x - 2y = z\sqrt{x^2+y^2+z^2}} \end{aligned}$$

5. (10 points) Label the following as reasonable or unreasonable:
   (a) $\mathbf{u}/\|\mathbf{v}\|$
   (b) $\mathbf{u}/\mathbf{v}$
   (c) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$
   (d) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$
   (e) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

Solution:

(a) Reasonable

(b) Unreasonable

(c) Unreasonable

(d) Either. If the outer dot is treated as a dot product, then unreasonable. If it is treated as a scalar product, then reasonable. (Students are not expected to tell the difference between $\cdot$ and $\bullet$)

(e) Reasonable