1. (20 points) Find the unit tangent vector, the principal unit normal vector, the binormal vector, and curvature for
$$\mathbf{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle.$$

Solution: Relevant formulas on cover page:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left\|\mathbf{T}'(t)\right\|}, \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \quad \kappa = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}$$

Tangent vector

$$\begin{aligned}
\mathbf{r}' &= \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle \\
\left\|\mathbf{r}'\right\| &= \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t} \\
\mathbf{T}(t) = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} &= \boxed{\left\langle \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{e^t}{e^t + e^{-t}}, \frac{-e^{-t}}{e^t + e^{-t}} \right\rangle}
\end{aligned}$$

Normal vector Doing quotient rule three times gives

$$\begin{aligned}
\mathbf{T}'(t) &= \left\langle \frac{-\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2} \right\rangle \\
\left\|\mathbf{T}'\right\| &= \sqrt{\frac{2(e^t - e^{-t})^2 + 8}{(e^t + e^{-t})^4}} = \sqrt{2}\sqrt{\frac{(e^{2t} - 2 + e^{-2t}) + 4}{(e^t + e^{-t})^4}} = \sqrt{2}\sqrt{\frac{(e^t + e^{-t})^2}{(e^t + e^{-t})^4}} = \frac{\sqrt{2}}{e^t + e^{-t}} \\
\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} &= \boxed{\left\langle -\frac{e^t - e^{-t}}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}} \right\rangle}
\end{aligned}$$

Binormal vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{bmatrix} \dfrac{\sqrt{2}}{e^t + e^{-t}} \\ \dfrac{e^t}{e^t + e^{-t}} \\ \dfrac{-e^{-t}}{e^t + e^{-t}} \end{bmatrix} \times \begin{bmatrix} -\dfrac{e^t - e^{-t}}{e^t + e^{-t}} \\ \dfrac{\sqrt{2}}{e^t + e^{-t}} \\ \dfrac{\sqrt{2}}{e^t + e^{-t}} \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{2}}{e^t + e^{-t}} \\ -\dfrac{2 - e^{-t}(e^t - e^{-t})}{(e^t + e^{-t})^2} \\ \dfrac{2 + e^t(e^t - e^{-t})}{(e^t + e^{-t})^2} \end{bmatrix}$$

$$= \left\langle \frac{\sqrt{2}}{e^t + e^{-t}},\ -\frac{1 + e^{-2t}}{(e^t + e^{-t})^2},\ \frac{1 + e^{2t}}{(e^t + e^{-t})^2} \right\rangle$$

$$= \boxed{\left\langle \frac{\sqrt{2}}{e^t + e^{-t}},\ -\frac{1}{e^t(e^t + e^{-t})},\ \frac{1}{e^{-t}(e^t + e^{-t})} \right\rangle}$$

Curvature

$$\begin{aligned}
\mathbf{r} &= \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle \\
\mathbf{r}' &= \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle \\
\mathbf{r}'' &= \left\langle 0, e^t, e^{-t} \right\rangle \\
\mathbf{r}' \times \mathbf{r}'' &= \begin{bmatrix} \sqrt{2} \\ e^t \\ -e^{-t} \end{bmatrix} \times \begin{bmatrix} 0 \\ e^t \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 1 - (-1) \\ -(\sqrt{2}e^{-t} - 0) \\ \sqrt{2}e^t \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2}e^{-t} \\ \sqrt{2}e^t \end{bmatrix} \\
\left\|\mathbf{r}' \times \mathbf{r}''\right\| &= \sqrt{4 + 2e^{-2t} + 2e^{2t}} = \sqrt{2}\sqrt{2 + e^{-2t} + e^{2t}} = \sqrt{2}(e^t + e^{-t}) \\
\left\|\mathbf{r}'\right\| &= \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}
\end{aligned}$$

so

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{2}(e^t + e^{-t})}{(e^t + e^{-t})^3} = \boxed{\frac{\sqrt{2}}{(e^t + e^{-t})^2}}$$

Alternatively,

$$\kappa = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|} = \frac{\sqrt{2}}{e^t + e^{-t}} \cdot \frac{1}{e^t + e^{-t}} = \boxed{\frac{\sqrt{2}}{(e^t + e^{-t})^2}}$$

2. (10 points) Let $f(x, y) = \ln(xy) - 1$, $y(u, v) = u^2 + v^2$, and $x(u, v) = uv$.
Find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$.

Solution:

$$\begin{tikzcd} & x \arrow[r] \arrow[ddr] & u \\ f \arrow[ur] \arrow[dr] & & \\ & y \arrow[r] \arrow[uur] & v \end{tikzcd}$$

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\
&= \frac{y}{xy} \cdot v + \frac{x}{xy} \cdot 2u \\
&= \frac{v}{x} + \frac{2u}{y} \\
&= \boxed{\frac{1}{u} + \frac{2u}{u^2 + v^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \\
&= \frac{y}{xy} \cdot u + \frac{x}{xy} \cdot 2v \\
&= \frac{u}{x} + \frac{2v}{y} \\
&= \boxed{\frac{1}{v} + \frac{2v}{u^2 + v^2}}
\end{aligned}$$

3. (10 points) Calculate the following:

(a) $\lim_{t\to 0} \left\langle \frac{e^t - 1}{t},\ \frac{3}{5-t^2},\ \sin(2t) \right\rangle$

Solution:

$$= \left\langle \lim_{t\to 0} \frac{e^t-1}{t}, \quad \lim_{t\to 0} \frac{3}{5-t^2}, \quad \lim_{t\to 0} \sin(2t) \right\rangle$$
$$= \left\langle \lim_{t\to 0} \frac{e^t}{1},\ \frac{3}{5},\ 0 \right\rangle = \boxed{\langle 1,\ -2,\ 0\rangle}$$

(b) The equation of the tangent plane to $f(x,y) = 2x^2 + xy - 6y^2$ at point $(3,-1)$.

Solution: Computing gives

$$f(3,-1) = 18 - 3 - 6 = 9$$

$$f_x(x,y) = 4x + y$$
$$f_x(3,-1) = 12 - 1 = 11$$

$$f_y(x,y) = x - 12y$$
$$f_y(3,-1) = 3 + 12 = 15$$

so

$$\boxed{z = 9 + 11(x-3) + 15(y+1)}$$

4. (20 points) Find all the first partial derivatives and second partial derivatives of

$$f(x, y) = \sqrt{x^2 + y} + 3x$$

Solution:

$$f_x = \frac{2x}{2\sqrt{x^2+y}} + 3 = \boxed{\frac{x}{\sqrt{x^2+y}} + 3}$$

$$f_y = \boxed{\frac{1}{2\sqrt{x^2+y}}}$$

$$f_{xx} = \frac{\sqrt{x^2+y} \cdot 1 - x \cdot \frac{2x}{2\sqrt{x^2+y}}}{x^2+y} = \frac{\sqrt{x^2+y} - \frac{x^2}{\sqrt{x^2+y}}}{x^2+y} = \frac{x^2+y-x^2}{(x^2+y)^{3/2}} = \boxed{\frac{y}{(x^2+y)^{3/2}}}$$

$$f_{xy} = x \cdot -\frac{1}{2}(x^2+y)^{-3/2} = \boxed{-\frac{x}{2(x^2+y)^{3/2}}}$$

$$f_{yx} = \frac{1}{2} \cdot -\frac{1}{2}(x^2+y)^{-3/2} \cdot 2x = \boxed{-\frac{x}{2(x^2+y)^{3/2}}}$$

$$f_{yy} = \frac{1}{2} \cdot -\frac{1}{2}(x^2+y)^{-3/2} = \boxed{-\frac{1}{4(x^2+y)^{3/2}}}$$

5. (15 points) Let

$$f(x,y,z) = xe^y + ye^z + ze^x.$$

(a) (5 points) Find the gradient of f.

Solution:

$$\boldsymbol{\nabla} f = \langle e^y + ze^x,\ xe^y + e^z,\ ye^z + e^x \rangle$$

(b) (10 points) Find $D_{\mathbf{u}} f(-1,2,0)$ in the direction of $\mathbf{v} = \langle -1,-1,-1 \rangle$.

Solution:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}} \langle -1,-1,-1 \rangle$$

$$\begin{aligned}
D_{\mathbf{u}} f(-1,2,0) &= \nabla f(-1,2,0) \cdot \mathbf{u} \\
&= \left\langle e^2 + 0e^{-1}, -e^2 + e^0, 2e^0 + e^{-1} \right\rangle \cdot \mathbf{u} \\
&= \left\langle e^2, -e^2 + 1, 2 + e^{-1} \right\rangle \cdot \mathbf{u} \\
&= \frac{1}{\sqrt{3}} \left(-e^2 - (-e^2 + 1) - (2 + e^{-1}) \right) \\
&= \boxed{\frac{1}{\sqrt{3}} \left(-3 - e^{-1} \right)} \\
&\approx -1.94445
\end{aligned}$$

6. (15 points) Calculate the limit if it exists. If the limit does not exist, explain why not.

(a) (5 points)

$$\lim_{(x,y)\to(1,-1)} \frac{5x-7y}{x+y+1}$$

Solution: Can just plug in:

$$\lim_{(x,y)\to(1,-1)} \frac{5x-7y}{x+y+1} = \frac{5+7}{1-1+1} = \boxed{12}$$

(b) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$

Solution: Approaching along the x-axis ($y=0$), the limit simplifies to

$$\lim_{x\to 0} \frac{0}{x^2+0} = 0$$

Approaching along $x = y^4$, the limit simplifies to

$$\lim_{y\to 0} \frac{y^8}{y^8+y^8} = \frac{1}{2}$$

Since approaching along two different paths gave two different results, the limit $\boxed{\text{does not exist}}$

7. (10 points) Consider $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, \frac{\sqrt{2}}{2}t^2, t \right\rangle$. Find the arc length function $s(t)$ for $\mathbf{r}(t)$. (Hint: What is $(t^2+1)^2$?)

Solution:

$$\mathbf{r}' = \left\langle t^2, \sqrt{2}t, 1 \right\rangle$$
$$\left\|\mathbf{r}'\right\| = \sqrt{t^4+2t^2+1} = \sqrt{(t^2+1)^2} = t^2+1$$

so

$$s(t) = \int_0^t \left\|\mathbf{r}'(u)\right\| \mathrm{d}u = \int_0^t (u^2+1)\,\mathrm{d}u = \frac{u^3}{3} + u \Big|_0^t = \boxed{\frac{t^3}{3}+t}$$