NAME Soln _______________ Rec. Instructor: _______________

Signature _______________ Rec. Time _______________

CALCULUS III - PRACTICE TEST 1

Show all work for full credit. No books or notes are permitted.

Problem	Points	Possible
1		20
2		20
3		30
4		20
5		10
Total Score		100

Note: Bold letters, like $\mathbf{u}$, are considered vectors unless specified otherwise.

You are free to use the following formulas on any of the problems.

Projection: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \underbrace{\left(\dfrac{u\cdot v}{\|u\|}\right)}_{\text{comp}_u v} \dfrac{u}{\|u\|}$

Cylindrical Coordinates:

$$x = r\cos(\theta) \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin(\theta) \qquad \tan(\theta) = \frac{y}{x}$$
$$z = z \qquad z = z$$

Spherical Coordinates:

$$x = \rho\cos(\theta)\sin(\varphi) \qquad \rho = \sqrt{x^2+y^2+z^2}$$
$$y = \rho\sin(\theta)\sin(\varphi) \qquad \tan(\theta) = \frac{y}{x}$$
$$z = \rho\cos(\varphi) \qquad \cos(\varphi) = \frac{z}{\rho}$$

(20) **1.** Define $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle -1, 0, 1\rangle$. Compute the following:

a) $\|\mathbf{u}\|$.
b) $\mathbf{u} \cdot \mathbf{v}$.
c) $\mathbf{u} \times \mathbf{v}$.
d) The area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.
e) The angle between $\mathbf{u}$ and $\mathbf{v}$.

a) $\sqrt{1+4+9} = \boxed{\sqrt{14}}$

b) $\vec{u} \cdot \vec{v} = (-1)(1) + 2(0) + 3(1) = -1 + 3 = \boxed{2}$

c) $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{vmatrix} = \langle 2, -(1+3), 2\rangle$

$= \boxed{\langle 2, -4, 2\rangle}$

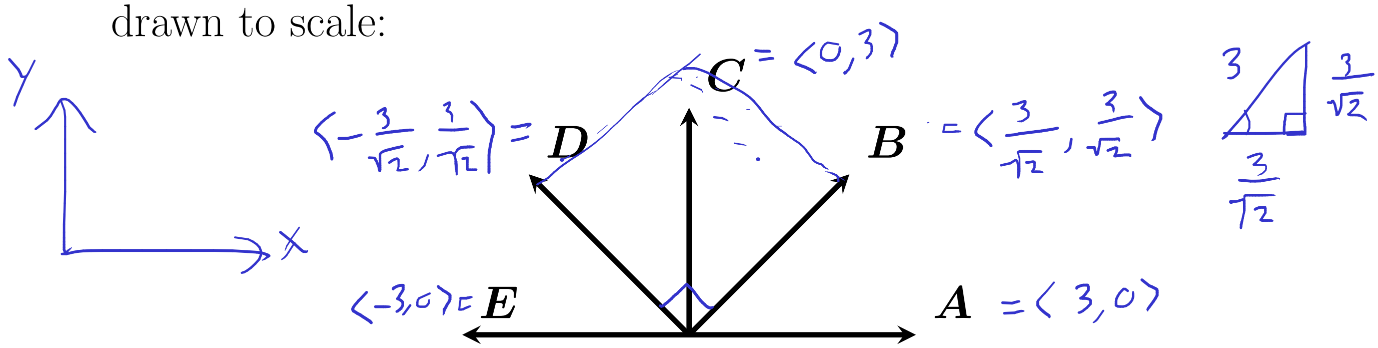
d) $\|\vec{u} \times \vec{v}\| = \|\langle 2, -4, 2\rangle\| = \sqrt{4+16+4} = \sqrt{24}$

$= \boxed{2\sqrt{6}}$

e) $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \dfrac{2}{\sqrt{14}\sqrt{2}} = \dfrac{2}{2\sqrt{7}} = \dfrac{1}{\sqrt{7}}$

$\theta = \boxed{\cos^{-1}\left(\frac{1}{\sqrt{7}}\right)}$

(20) **2.** For this problem we refer to the following diagram, which is drawn to scale:



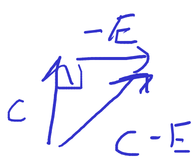
The vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$ all have length three. All of the angles between the vectors are multiples of 45 degrees. Compute the following explicitly:

a) $\mathbf{A} \cdot \mathbf{E}$
b) $\|\mathbf{B} \times \mathbf{D}\|$
c) $\mathbf{B} \cdot \mathbf{C}$
d) $\|\mathbf{C} - \mathbf{E}\|$
e) $\mathbf{A} \cdot \mathbf{A}$

a) $\vec{A} \cdot \vec{E} = \|\vec{A}\| \cdot \|\vec{E}\| \cos \pi$
$= 9(-1) = \boxed{-9}$

b) $3 \cdot 3 = \boxed{9}$

c) $\vec{B} \cdot \vec{C} = \|\vec{B}\| \, \|\vec{C}\| \cos \frac{\pi}{4}$
$= 9 \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{9\sqrt{2}}{2}}$

d) 

$\|\vec{C} - \vec{E}\| = \sqrt{9+9} = \boxed{3\sqrt{2}}$

e) $A \cdot A = \|A\|^2 = 3^2 = \boxed{9}$

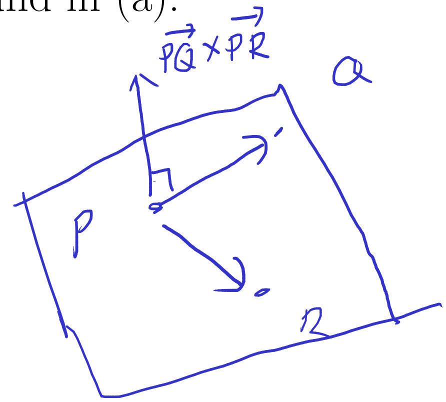
(30) **3.**

a) Find an equation for the plane containing the points $P = (1, 1, 1)$, $Q = (3, 2, 0)$, and $R = (2, 0, 1)$. Express your answer in the form $Ax + By + Cz = D$.

b) Find the shortest distance from point $S = (-1, 9, 1)$ to the plane you found in (a).

c) Find the equation for the line passing through point Q and perpendicular to the plane you found in (a).

$T = (x, y, z)$



a)

$$\vec{n} \cdot \vec{PT} = 0$$

$$\vec{PQ} = \langle 2, 1, -1 \rangle$$

$$\vec{PR} = \langle 1, -1, 0 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \langle -1, -(0+1), -3 \rangle$$

$$= \langle -1, -1, -3 \rangle$$

$$\vec{n} = \langle 1, 1, 3 \rangle$$

$$\langle 1, 1, 3 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$$

$$x - 1 + y - 1 + 3(z-1) = 0$$

$$\boxed{x + y + 3z = 5}$$

(30) **3.**

a) Find an equation for the plane containing the points $P = (1, 1, 1)$, $Q = (3, 2, 0)$, and $R = (2, 0, 1)$. Express your answer in the form $Ax + By + Cz = D$.

b) Find the shortest distance from point $S = (-1, 9, 1)$ to the plane you found in (a).

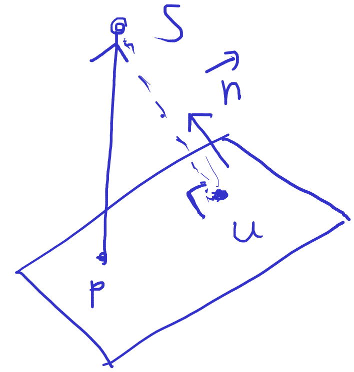
c) Find the equation for the line passing through point Q and perpendicular to the plane you found in (a).

b)
$$\|\mathrm{proj}_{\vec{n}} \vec{PS}\| = |\mathrm{comp}_{\vec{n}} \vec{PS}|$$

$$= \frac{|\vec{PS} \cdot \vec{n}|}{\|\vec{n}\|}$$

$$= \frac{|-2+8+0|}{\sqrt{1+1+9}}$$

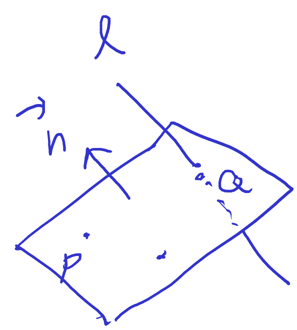
$$= \boxed{\frac{6}{\sqrt{11}}}$$



$\vec{PS} = \langle -2, 8, 0 \rangle$

$\vec{n} = \langle 1, 1, 3 \rangle$

c)
$$\boxed{\begin{aligned} \ell &= \langle 3, 2, 0 \rangle + t\langle 1, 1, 3 \rangle \quad t \in \mathbb{R} \\ &= \langle 3+t, 2+t, 3t \rangle \end{aligned}}$$



(20) **4.** Convert the equation written in spherical coordinates into an equation in Cartesian coordinates.

$$\frac{1}{\sin\varphi} = \csc(\varphi) = 2\cos(\theta) + 4\sin(\theta)$$

$x = \rho\cos(\theta)\sin(\varphi)$ $\qquad$ $\rho = \sqrt{x^2+y^2+z^2}$

$y = \rho\sin(\theta)\sin(\varphi)$ $\qquad$ $\tan(\theta) = \frac{y}{x}$

$z = \rho\cos(\varphi)$ $\qquad$ $\cos(\varphi) = \frac{z}{\rho}$

$$1 = 2\cos\theta\sin\varphi + 4\sin\theta\sin\varphi$$

$$\rho = 2\rho\cos\theta\sin\varphi + 4\rho\sin\theta\sin\varphi$$

$$\boxed{\sqrt{x^2+y^2+z^2} = 2x + 4y}$$

r u

(10) **5.** Label the following as reasonable or unreasonable:

a) $2/\mathbf{v}$ u

b) $\mathbf{u}/2$ r

c) $\mathbf{u} \cdot \mathbf{v} = 2$ r

d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ r

e) $3 \times \mathbf{v}$ u