1. Find the unit tangent vector, the principal unit normal vector, the binormal vector, and curvature for
$$\mathbf{r}(t) = \left\langle 3t, \cos(4t), \sin(4t) \right\rangle .$$

Solution: Relevant formulas on cover page:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left\|\mathbf{T}'(t)\right\|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \qquad \kappa = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|}$$

Tangent vector

$$\mathbf{r}' = \left\langle 3, -4\sin(4t), 4\cos(4t) \right\rangle$$
$$\left\|\mathbf{r}'\right\| = \sqrt{9 + 16\sin^2(4t) + 16\cos^2(4t)} = \sqrt{9+16} = 5$$
$$\mathbf{T}(t) = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \boxed{\left\langle \frac{3}{5}, -\frac{4}{5}\sin(4t), \frac{4}{5}\cos(4t) \right\rangle}$$

Normal vector

$$\mathbf{T}'(t) = \left\langle 0, -\frac{16}{5}\cos(4t), -\frac{16}{5}\sin(4t) \right\rangle$$
$$\left\|\mathbf{T}'\right\| = \sqrt{\left(-\frac{16}{5}\right)^2} = \frac{16}{5}$$
$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \boxed{\left\langle 0, -\cos(4t), -\sin(4t) \right\rangle}$$

Binormal vector

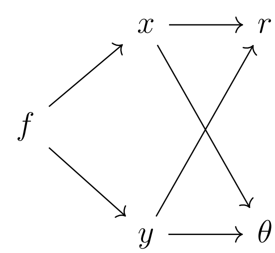
$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \tfrac{3}{5} & -\tfrac{4}{5}\sin(4t) & \tfrac{4}{5}\cos(4t) \\ 0 & -\cos(4t) & -\sin(4t) \end{vmatrix}$$
$$= \left[\frac{4}{5}\sin^2(4t) + \frac{4}{5}\cos^2(4t)\right]\mathbf{i} - \left[-\frac{3}{5}\sin(4t)\right]\mathbf{j} + \left[-\frac{3}{5}\cos(4t)\right]\mathbf{k}$$
$$= \boxed{\left\langle \frac{4}{5}, \frac{3}{5}\sin(4t), -\frac{3}{5}\cos(4t) \right\rangle}$$

Curvature

$$\kappa = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|} = \frac{16/5}{5} = \boxed{\frac{16}{25}}$$

2. Let $f(x,y) = x^2 - xy + 3y^2$, $y(r,\theta) = r\sin(\theta)$, and $x(r,\theta) = r\cos(\theta)$.
Find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$.

Solution:



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$= \boxed{(2x-y)\cos\theta + (-x+6y)\sin\theta}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$
$$= \boxed{(2x-y)(-r\sin\theta) + (-x+6y)r\cos\theta}$$

3. Calculate the following:

(a) $\lim_{t\to\infty} \left\langle \frac{\ln(t)}{t^2},\ \frac{2t^2}{1-t-t^2},\ e^{-t} \right\rangle$

Solution:

$$= \left\langle \lim_{t\to\infty} \frac{\ln(t)}{t^2}, \quad \lim_{t\to\infty} \frac{2t^2}{1-t-t^2}, \quad \lim_{t\to\infty} e^{-t} \right\rangle$$
$$= \left\langle \lim_{t\to\infty} \frac{1/t}{2t},\ -2,\ 0 \right\rangle = \boxed{\langle 0,\ -2,\ 0\rangle}$$

(b) The equation of the tangent plane to $f(x, y) = x^2y - \sqrt{x+y}$ at point $(1, 2)$.

Solution: Computing gives

$$f(1,2) = 2 - \sqrt{3}$$
$$f_x = 2xy - \frac{1}{2\sqrt{x+y}}$$
$$f_x(1,2) = 4 - \frac{1}{2\sqrt{3}}$$
$$f_y = x^2 - \frac{1}{2\sqrt{x+y}}$$
$$f_y(1,2) = 1 - \frac{1}{2\sqrt{3}}$$

so

$$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$$
$$\boxed{z = 2 - \sqrt{3} + \left(4 - \frac{1}{2\sqrt{3}}\right)(x-1) + \left(1 - \frac{1}{2\sqrt{3}}\right)(y-2)}$$

4. Find all the first partial derivatives and second partial derivatives of

$$f(x,y) = xy^2\ln(x) + 3\cos(x).$$

Solution:

$$f_x = y^2(\ln x + 1) - 3\sin x$$
$$f_y = 2xy\ln x$$

$$f_{xx} = \frac{y^2}{x} - 3\cos x$$
$$f_{xy} = 2y(\ln x + 1)$$
$$f_{yx} = 2y(\ln x + 1)$$
$$f_{yy} = 2x\ln x$$

5. Calculate the limit if it exists. If the limit does not exist, explain why not.

(a) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{-ye^x}{x+y^2}$

Solution: Can just plug in:

$$\lim_{(x,y)\to(1,2)} \frac{-ye^x}{x+y^2} = \boxed{\frac{-2e}{5}}$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$

Solution: Just plugging in yields $\frac{0}{0}$, so we must try something else. Approaching along the x-axis ($y = 0$), the limit simplifies to

$$\lim_{x\to 0} \frac{0}{x^6} = 0$$

Approaching along the curve $y = x^3$, the limit simplifies to

$$\lim_{x\to 0} \frac{x^3 \cdot x^3}{x^6 + x^6} = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$$

Thus, $\boxed{\text{the limit does not exist}}$

6. Let

$$f(x, y, z) = x^2y + y^2z + z^2x.$$

(a) Find the gradient of f.

Solution:

$$\nabla f = \boxed{\left\langle 2xy + z^2,\ x^2 + 2yz,\ y^2 + 2zx \right\rangle}$$

(b) Find $D_{\mathbf{u}}f(1, 1, 1)$ in the direction of $\mathbf{v} = \left\langle \sqrt{2}, \sqrt{2}, \sqrt{2} \right\rangle$.

Solution: First, notice that $\|\mathbf{v}\| = \sqrt{2+2+2} = \sqrt{6}$, so we must normalize $\mathbf{v}$ before computing the directional derivative:

$$\mathbf{u} = \hat{\mathbf{v}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

Then,

$$\begin{aligned} D_{\mathbf{u}}f(1,1,1) &= \nabla f(1,1,1) \cdot \mathbf{u} \\ &= \langle 3, 3, 3 \rangle \cdot \mathbf{u} \\ &= \boxed{3\sqrt{3}} \end{aligned}$$

7. Consider $\mathbf{r}(t) = \big\langle 2t, 3\cos(2t), 3\sin(2t)\big\rangle$.

(a) Find the arc length function $s(t)$ for $\mathbf{r}(t)$.

Solution: Recall the formula:

$$s(t) = \int_0^t \big\|\mathbf{r}'(u)\big\| \, \mathrm{d}u$$

so we compute

$$\mathbf{r}' = \langle 2, -6\sin 2t, 6\cos 2t\rangle$$
$$\big\|\mathbf{r}'\big\| = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

therefore

$$s(t) = \int_0^t 2\sqrt{10}\, \mathrm{d}u$$
$$\boxed{s(t) = 2\sqrt{10}t}$$

(b) Find the arc length parametrization $\mathbf{r}(s)$.

Solution: The previous part gave the relation between s and t:

$$s = 2\sqrt{10}t \implies t = \frac{s}{2\sqrt{10}} =: t(s)$$

The arclength parametrization is thus

$$\mathbf{r}(s) = \mathbf{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}},\ 3\cos\left(\frac{s}{\sqrt{10}}\right),\ 3\sin\left(\frac{s}{\sqrt{10}}\right)\right\rangle$$