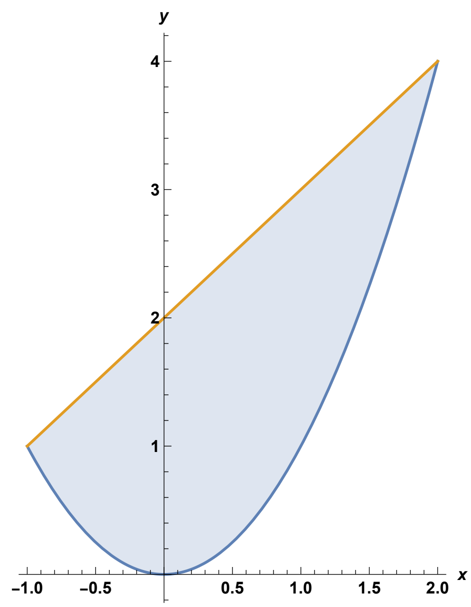
1. Integrate $f(x, y) = x$ over the region bounded by $y = x^2$ and $y = x + 2$.

Solution: A quick sketch of this region can be determined:



Calling the region to be integrated over D, the integral is thus

$$\begin{aligned}\iint_D f(x, y)\,\mathrm{d}A &= \int_{-1}^{2}\int_{x^2}^{x+2} x\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-1}^{2} x\,[y]_{y=x^2}^{x+2}\,\mathrm{d}x \\ &= \int_{-1}^{2} x(x+2-x^2)\,\mathrm{d}x \\ &= \int_{-1}^{2} (-x^3+x^2+2x)\,\mathrm{d}x \\ &= \boxed{\frac{9}{4}}\end{aligned}$$

2. Let R be the region given by $x \geq 0$, $y \leq 0$, and $x^2 + y^2 \leq z \leq 4$.

   (a) If you were integrating over R, would you integrate the region in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you chose the coordinate system you did, and express the bounds for the region R in that coordinate system.

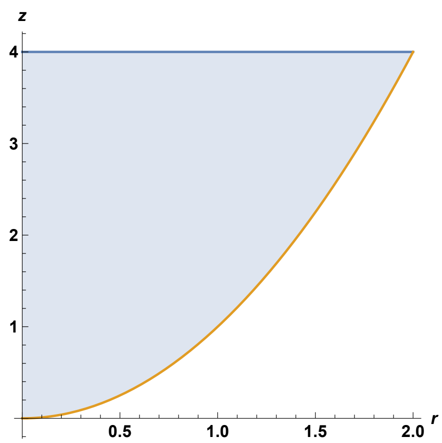
Solution: Manipulating the bounding inequality,

$$x^2 + y^2 \leq z \leq 4 \implies 0 \leq r^2 \leq z \leq 4.$$

From this we can recognize that the region R is radially symmetric. The inequalities $x \geq 0$ and $y \leq 0$ specify what values of θ are allowed. Namely, θ must be in quadrant IV, so $\theta \in [-\frac{\pi}{2}, 0]$

Coming back to the series of inequalities $0 \leq r^2 \leq z \leq 4$, they can be read as saying that z is bounded above by the value 4 and bounded below by the function r^2. This allows us to draw a cross section of R in the rz-plane:

Figure 1: Cross section of R in the rz-plane



The region R is the solid of revolution obtained by revolving this cross section about the z-axis.

Setting up an integral over this region in Cartesian or Spherical is possible, but bothersome. Setting it up in Cylindraical is most natural.

We can clearly see from the cross section that $r \in [0, 2]$ and $z \in [r^2, 4]$.

The region R can be expressed as

$$\boxed{R = \{(r, \theta, z) \mid -\frac{\pi}{2} \leq \theta \leq 0,\ 0 \leq r \leq 2,\ r^2 \leq z \leq 4\}}$$

(b) Find the volume of R

Solution: The volume of R can be computed as

$$\begin{aligned}
\iiint_R 1\,\mathrm{d}V &= \int_{-\pi/2}^{0}\int_0^2\int_{r^2}^{4} 1\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{-\pi/2}^{0} 1\,\mathrm{d}\theta \cdot \int_0^2\int_{r^2}^{4} r\,\mathrm{d}z\,\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^2 r\,[z]_{z=r^2}^{4}\,\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^2 (4r - r^3)\,\mathrm{d}r \\
&= \frac{\pi}{2}\cdot\left[2r^2 - \frac{r^4}{4}\right]_0^2 = \boxed{2\pi}
\end{aligned}$$

3. Find and classify all the critical points of $f(x, y) = 3x^2y + -3x^2 - 3y^2 + 2$.

Solution: Solving $\boldsymbol{\nabla} f = \mathbf{0}$ finds the critical points of f:

$$\boldsymbol{\nabla} f = \langle f_x, f_y \rangle = \langle 6xy - 6x, 3x^2 + 3y^2 - 6y \rangle = \mathbf{0}$$
$$\implies \begin{cases} 6x(y-1) = 0 \\ 3x^2 + 3y^2 - 6y = 0 \end{cases}$$

The first equation implies $x = 0$ or $y = 1$.

- If $x = 0$, then the second equation reduces to

  $$3y^2 - 6y = 0 \implies 3y(y-2) = 0 \implies y = 0, 2$$

  This yields the critical points $(0, 0)$ and $(0, 2)$.

- If $y = 1$, then the second equation reduces to

  $$3x^2 - 3 = 0 \implies 3(x^2 - 1) = 0 \implies x = \pm 1$$

  This yields the critical points $(1, 1)$ and $(-1, 1)$.

Now we test these four critical points. First, we compute the discriminant:

$$\begin{aligned} f_{xx} &= 6y - 6 = 6(y-1) \\ f_{yy} &= 6y - 6 = 6(y-1) \\ f_{xy} &= 6x \\ \implies D &= f_{xx}f_{yy} - f_{xy}^2 = 36(y-1)^2 - 36x^2 \end{aligned}$$

Using the second derivative test:

$$\begin{aligned} &D(0,0) = 36 > 0 \quad \text{and} \quad f_{xx}(0,0) = -6 < 0 && \implies (0,0) \text{ is a local max} \\ &D(0,0) = 36 > 0 \quad \text{and} \quad f_{xx}(0,0) = 6 > 0 && \implies (0,0) \text{ is a local min} \\ &D(1,1) = -36 < 0 && \implies (1,1) \text{ is a saddle point} \\ &D(-1,1) = -36 < 0 && \implies (-1,1) \text{ is a saddle point} \end{aligned}$$

4. Use Lagrange Multipliers to find the maximum and minimum of the following function

$$f(x,y,z) = x + y^2 - z$$

subject to the constraint

$$g(x,y,z) = x^2 + y^2 + z^2 = 1.$$

Solution:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \implies \begin{cases} \langle 1, 2y, -1\rangle = \lambda \langle 2x, 2y, 2z\rangle \\ x^2+y^2+z^2 = 1 \end{cases} \implies \begin{cases} 1 = 2\lambda x \\ 2y = 2\lambda y \\ -1 = 2\lambda z \\ x^2+y^2+z^2 = 1 \end{cases}$$

Manipulating the second equation:

$$2y = 2\lambda y \implies 2y(\lambda - 1) = 0$$

Thus either $y = 0$ or $\lambda = 1$.

- If $\lambda = 1$, the first equation implies $x = \frac{1}{2}$ and the third equation implies $z = -\frac{1}{2}$. Substituting these into the constraint equation gives

$$y^2 = \frac{1}{2} \implies y = \pm\frac{1}{\sqrt{2}}$$

This thus gives two constrained critical points: $(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2})$

- If $y = 0$ (and we assume $\lambda \neq 0$, which is reasonable), the system reduces to

$$\begin{cases} x = \dfrac{1}{2\lambda} \\ z = -\dfrac{1}{2\lambda} \\ x^2 + z^2 = 1 \end{cases} \implies x^2 + (-x)^2 = 1 \implies 2x^2 = 1 \implies x = \pm\frac{1}{\sqrt{2}}$$

This results in two constrained critical points: $(\pm\frac{1}{\sqrt{2}}, 0, \mp\frac{1}{\sqrt{2}})$

We now evaluate f on all of the constrained critical points that we found:

$$f(x,y,z) = x + y^2 - z$$
$$f\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} - \left(-\frac{1}{2}\right) = \frac{3}{2}$$
$$f\left(\pm\frac{1}{\sqrt{2}}, 0, \mp\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}} + 0 - \left(\mp\frac{1}{\sqrt{2}}\right) = \pm\frac{2}{\sqrt{2}} = \pm\sqrt{2}$$

Thus, the minimum value is $-\sqrt{2}$ and the maximum value is $\frac{3}{2}$.

5. Let E be the region inside $x^2 + y^2 + z^2 = 25$ and $z = -\sqrt{3x^2 + 3y^2}$, and where $y \geq 0$.

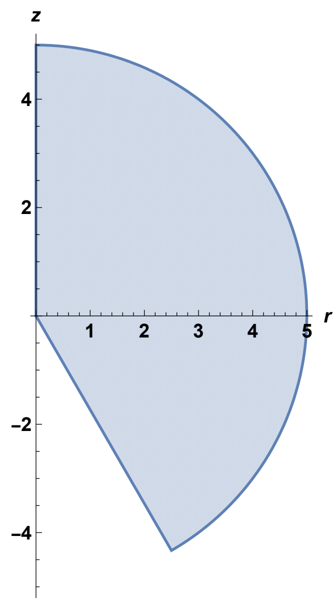
   (a) If you were integrating over E, would you integrate the region in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you chose the coordinate system you did, and express the bounds for the region E in that coordinate system.

Solution: Manipulating the bounding functions:

$$x^2 + y^2 + z^2 = 25 \rightsquigarrow \rho^2 = 25 \rightsquigarrow r^2 + z^2 = 25$$
$$z = -\sqrt{3x^2 + 3y^2} = -\sqrt{3}\sqrt{x^2 + y^2} = -\sqrt{3}r$$

We can notice that the region E is radially symmetric. Using the equations $r^2 + z^2 = 25$ and $z = -\sqrt{3}r$, we can plot the cross section of E:

Figure 2: Cross section of E in the rz-plane



The region E is the solid of revolution generated by revolving this cross section around the z-axis. The condition $y \geq 0$, specifies what angles θ are valid. In particular, this inequality says that θ is in quadrants I and II ($\theta \in [0, \pi]$).

Trying to integrate over this region in Cartesian or Cylindrical would be bothersome. The most natural choice would be to use Spherical.

The bounds for ρ is clear: $\rho \in [0, 5]$.

To determine the bounds for φ, we need the angle for the diagonal line in the cross section. We thus solve

$$\begin{aligned}
& z = -\sqrt{3}\sqrt{x^2+y^2} \\
\implies \quad & \rho\cos\varphi = -\sqrt{3}\sqrt{\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi} \\
\implies \quad & \rho\cos\varphi = -\sqrt{3}\sqrt{\rho^2\sin^2\varphi} = -\sqrt{3}\rho\sin\varphi \\
\implies \quad & -\frac{1}{\sqrt{3}} = \tan\varphi \\
\implies \quad & \varphi = \frac{5\pi}{6}
\end{aligned}$$

Hence $\varphi \in [0, \frac{5\pi}{6}]$.
(Note that a calculator will give $\tan^{-1}(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$ since the range of arctan is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This is not the correct angle, since in spherical, $\varphi \in [0, \pi]$. We want the version of the angle in quadrant II, not quadrant IV as the function arctan gives.)
The description for E in Spherical is thus

$$\boxed{E = \{(\rho, \varphi, \theta) \mid 0 \leq \rho \leq 5,\ 0 \leq \varphi \leq \frac{5\pi}{6},\ 0 \leq \theta \leq \pi\}}$$

(b) Evaluate

$$\iiint_E x \, \mathrm{d}V.$$

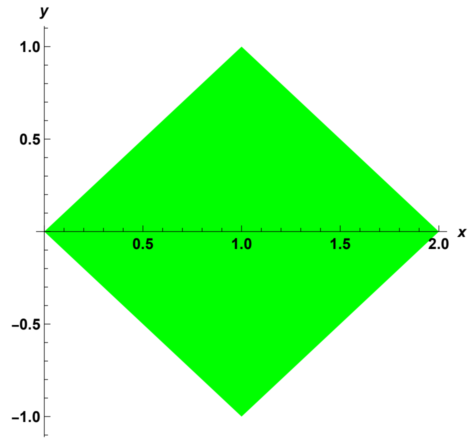
Solution:

$$\begin{aligned}
&= \int_0^{\pi}\int_0^{5\pi/6}\int_0^5 \rho\cos\theta\sin\varphi \cdot \rho^2\sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}\theta \\
&= \int_0^5 \rho^3 \, \mathrm{d}\rho \cdot \int_0^{5\pi/6} \sin^2\varphi \, \mathrm{d}\varphi \cdot \underbrace{\int_0^{\pi} \cos\theta \, \mathrm{d}\theta}_{=0} \\
&= \boxed{0}
\end{aligned}$$

6. Evaluate
$$\iint_R (x+y)\,\mathrm{d}A$$
where R is the region with vertices given by the points $(0,0)$, $(2,0)$, $(1,1)$, and $(1,-1)$ using the transformation $x(u,v) = 2u+3v$ and $y(u,v) = 2u-3v$.

Solution:

Figure 3: The region R in the xy-plane

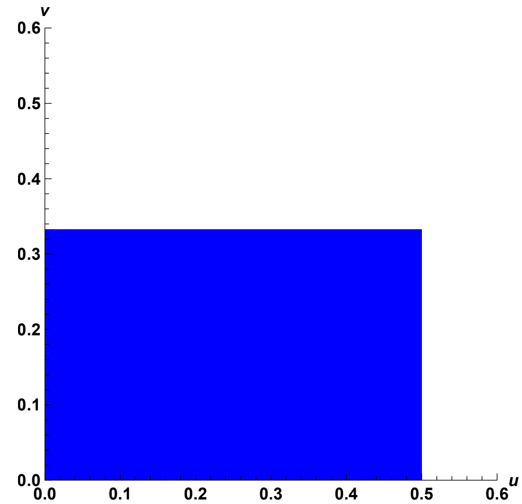


The transformation is given in the form $T : (u,v) \to (x,y)$. In order to determine the preimage of this region in the uv-plane, we need to construct the inverse transformation:

$$T : \begin{cases} x = 2u+3v \\ y = 2u-3v \end{cases} \implies \begin{cases} x+y = 4u \\ x-y = 6v \end{cases} \implies T^{-1} : \begin{cases} u = \dfrac{1}{4}(x+y) \\ v = \dfrac{1}{6}(x-y) \end{cases}$$

Using this inverse transformation, we get the points:

Figure 4: The region S in the uv-plane



$$
\begin{aligned}
T^{-1}(x,y) &= (\tfrac{1}{4}(x+y), \tfrac{1}{6}(x-y)) = (u,v) \\
T^{-1}(0,0) &= (0,0) \\
T^{-1}(2,0) &= (\tfrac{1}{2}, \tfrac{1}{3}) \\
T^{-1}(1,1) &= (\tfrac{1}{2}, 0) \\
T^{-1}(1,-1) &= (0, \tfrac{1}{3})
\end{aligned}
$$

These points define the region S in the uv-plane, which gets maps to the region R in the xy-plane under the transformation T.

Setting up the integral over the region S is easy.

To convert the given integral from the xy-plane to the uv-plane, we will also need to compute the Jacobian of the transformation:

$$\text{Jac}(T) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6 - 6 = -12$$

Thus we have (by the Change of Variables formula):

$$
\begin{aligned}
\iint_R (x+y)\,\mathrm{d}A &= \iint_S 4u\big|\text{Jac}(T)\big|\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^{1/3}\int_0^{1/2} 4u \cdot 12\,\mathrm{d}u\,\mathrm{d}v \\
&= 48\int_0^{1/2} u\,\mathrm{d}u \cdot \int_0^{1/3} 1\,\mathrm{d}v \\
&= \frac{48}{3}\left[\frac{u^2}{2}\right]_0^{1/2} \\
&= 16 \cdot \frac{1}{8} = \boxed{2}
\end{aligned}
$$

7. Answer the following short answer questions.

(a) Give an example of a region that is Type 2, but not Type 1. Write the region explicitly, don't just graph it.

Solution: Answers vary.

(b) Write out the Extreme Value Theorem (Theorem 4.18 from the textbook).

Solution: Suppose $f(x, y)$ is a continuous function on a closed and bounded domain D. Then f attains its minimum and minimum on D.

(c) If the volume of a region E is 2, and $\iiint_E f(x, y, z)\,\mathrm{d}V = 3$, what is the average value of $f(x, y, z)$?

Solution: In general:

$$f_{\text{avg}} = \frac{\iiint_E f\,\mathrm{d}V}{\iiint_E 1\,\mathrm{d}V} = \frac{\iiint_E f\,\mathrm{d}V}{\text{vol}(E)}$$

Thus for this problem,

$$f_{\text{avg}} = \boxed{\frac{3}{2}}$$

(d) If I set up an integral that looks like:

$$\int_0^1 \int_0^z \int_0^{x+y} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

what is the problem with my integral set up?

Solution: The bounds on the z-integral says that z depends on both x and y. The bounds on the y integral says that y depends on z. This results in a cyclic dependancy, as z depends on y and y depends on z. Visually:

$$z \rightleftarrows y$$

This is not allowed in integral setups.

Another way of thinking about this:

The variables that are used as bounds of an integral must be determined by things *outside* of the integral. Valid examples of integrals:

$$\int_0^1 \int_{f(x)}^{g(x)} h(x, y)\,\mathrm{d}y\,\mathrm{d}x, \qquad F(t) = \int_0^t f(x)\,\mathrm{d}x$$

In the left example, the bounds of the y-integral depend on x, which is fine, as x is fixed outside of the y-integral (by the x-integral surrounding it). In the right example, the x-integral depends on t which is part of the outer context of the expression. The overall integral can be thought of as a function whose value depends on t.

Contrast this with the integral given. The y-integral requires the variable z, which appears to be reaching *into* the inner z-integral. This is not allowed.

(e) Suppose we want to integrate

$$\iint_R y\sin(xy)\,\mathrm{d}A$$

over $R = [1,2]\times[0,\pi]$. Which is less steps: integrating with respect to x first, or y first? Why?

Solution: Integrating with respect to x is easier, as the factor y gets absorbed by the integration (reversing chain rule):

$$\int_1^2 y\sin(xy)\,\mathrm{d}x = -\cos(xy)\Big|_{x=1}^{2} = \cos(y) - \cos(2y)$$

The result is easy to then integrate with respect to y.

If we tried to integrate with respect to y *first*, the integral would require integration by parts.

$$\begin{aligned}\int_0^\pi y\sin(xy)\,\mathrm{d}y &= -\frac{y}{x}\cos(xy)\Big|_{y=0}^{\pi} + \int_0^\pi \frac{\cos(xy)}{x}\,\mathrm{d}y \\ &= -\frac{y}{x}\cos(xy)\Big|_{y=0}^{\pi} + \frac{\sin(xy)}{x^2}\Big|_{y=0}^{\pi} \\ &= \frac{\sin(\pi x) - \pi x\cos(\pi x)}{x^2}\end{aligned}$$

Trying to integrate the result with respect to x would also be difficult, though possible. Both $\int \frac{\cos x}{x}\,\mathrm{d}x$ and $\int \frac{\sin x}{x^2}\,\mathrm{d}x$ cannot be expressed as elementary functions. Instead, one would have to recognize the integrand as coming from an application of the quotient rule, and reverse the quotient rule:

$$\int \frac{\sin(\pi x) - \pi x\cos(\pi x)}{x^2}\,\mathrm{d}x = -\frac{\sin \pi x}{x}$$