Exam 3 soln

Friday, July 14, 2023 2:44 AM

NAME Soln

Rec. Instructor:

Signature

Rec. Time

CALCULUS III - EXAM 3

Show all work for full credit. No books or notes are permitted.

Problem	Points	Possible
1		15
2		20
3		10
4		20
5		20
6		15
Total Score		100

Note: Bold letters, like **u**, are considered vectors unless specified otherwise.

You are free to use the following formulas on any of the problems.

Cylindrical Coordinates:

$$x = r\cos(\theta) \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin(\theta) \qquad \tan(\theta) = \frac{y}{x}$$

$$z = z \qquad z = z$$

$$dV = r\,dr\,d\theta\,dz$$

Spherical Coordinates:

$$x = \rho\cos(\theta)\sin(\varphi) \qquad \rho = \sqrt{x^2+y^2+z^2}$$

$$y = \rho\sin(\theta)\sin(\varphi) \qquad \tan(\theta) = \frac{y}{x}$$

$$z = \rho\cos(\varphi) \qquad \cos(\varphi) = \frac{z}{\rho}$$

$$dV = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

Second Derivative Test: Let $z = f(x,y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

Change of Variables: We have $T : S \to R$ and $T(u,v) = (g(u,v), h(u,v))$.
$\text{Jac}(T) = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right|$ and

$$\int\int_S f(x,y)\,dx\,dy = \int\int_S f(g(u,v), h(u,v)) |\text{Jac}(T)|\,du\,dv.$$

1. (15 points) Find and classify all the critical points of

$$f(x,y) = 2x^3 - 6xy + y^2.$$

$$\nabla f = \langle \underbrace{6x^2 - 6y}_{f_x}, \underbrace{-6x + 2y}_{f_y} \rangle = \vec{0}$$

$$\Rightarrow \begin{cases} 6x^2 - 6y = 6(x^2 - y) = 0 \\ -6x + 2y = 2(-3x + y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y = x^2 \\ y = 3x \end{cases}$$

$$x^2 = 3x \Rightarrow x^2 - 3x = x(x-3) = 0$$

$x = 0 \Rightarrow y = 0$ Crit points: $(0,0)$, $(3,9)$

$x = 3 \Rightarrow y = 9$

$f_{xx} = 12x$

$f_{yy} = 2$

$f_{xy} = -6$

$$D = f_{xx} f_{yy} - f_{xy}^2$$
$$= 24x - 36$$
$$= 12(2x - 3)$$

$D(0,0) = -36 < 0 \Rightarrow \boxed{(0,0) \text{ saddle pt}}$

$D(3,9) = 36 > 0$, $f_{xx}(3,9) = 36 > 0$

$\Rightarrow \boxed{(3,9) \text{ local min}}$

2. (20 points) Use Lagrange Multipliers to find the maximum and minimum of the following function

$$f(x, y, z) = x - 2y + 2z$$

subject to the constraint

$$\frac{x^2}{4} + y^2 + z^2 = 12.$$

$\nabla f = \lambda \nabla g$

$\langle 1, -2, 2 \rangle = \lambda \langle \frac{x}{2}, 2y, 2z \rangle$

$$\begin{cases} 1 = \frac{\lambda}{2} x & \Rightarrow x = \frac{2}{\lambda} \\ -2 = 2\lambda y & \Rightarrow y = \frac{-1}{\lambda} \\ 2 = 2\lambda z & \Rightarrow z = \frac{1}{\lambda} \\ \frac{x^2}{4} + y^2 + z^2 = 12 \end{cases}$$

$y = -z$

$2z = x$

$$\frac{(2z)^2}{4} + (-z)^2 + z^2 = 3z^2 = 12$$

$$\Rightarrow z^2 = 4$$

$$\Rightarrow z = \pm 2$$

crit points:

$z = 2$: $(4, -2, 2) \longrightarrow f(4, -2, 2) = 4 + 4 + 4 = 12 \leftarrow$ max

$z = -2$: $(-4, 2, -2) \longrightarrow f(-4, 2, -2) = -4 - 4 - 4 = -12 \leftarrow$ min

3. (10 points) Calculate the integral

$$\int\int_B xe^y dA$$

on $B = [1, 2] \times [0, 3]$.

$$\int_0^3 \int_1^2 xe^y \, dx \, dy = \int_0^3 e^y \, dy \cdot \int_1^2 x \, dx$$

$$= e^y\Big]_0^3 \cdot \frac{x^2}{2}\Big]_1^2$$

$$= (e^3 - 1) \cdot \frac{1}{2}(4-1)$$

$$= \boxed{\frac{3}{2}(e^3 - 1)}$$

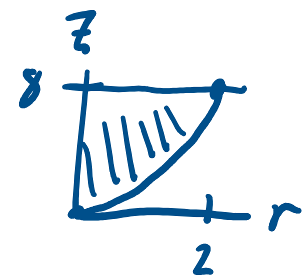
$2r^2 = z$

4. Let E be the region bounded below by $2x^2 + 2y^2 = z$ and above by the plane $z = 8$. Consider

$$\int\int\int_E x^2 + y^2 dV.$$

a) (8 points) If you were evaluating the given integral over E, would you integrate in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you choose the coordinate system you did, and express the bounds for the region E in that system.

radial symtry, flat top $\rightsquigarrow$ use cyl.



$$\boxed{\begin{aligned} &\theta \in [0, 2\pi] \\ &r \in [0, 2] \\ &z \in [2r^2, 8] \end{aligned}}$$

b) (12 points) Evaluate

$$\int\int\int_E x^2 + y^2 dV.$$

$$= \int_0^{2\pi}\int_0^{2}\int_{2r^2}^{8} r^2 \cdot r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^2 \int_{2r^2}^{8} r^3 \, dz \, dr$$

$$= 2\pi \int_0^2 r^3(8 - 2r^2) \, dr = 2\pi\left[2r^4 - \frac{1}{3}r^6\right]_0^2$$

$$= 8r^3 - 2r^5$$

$$= 2\pi\left[32 - \frac{64}{3}\right]$$

$$= 64\pi\left[1 - \frac{2}{3}\right] = \boxed{\frac{64\pi}{3}}$$

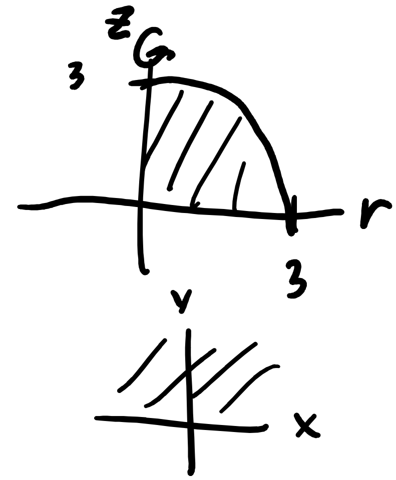
$$= 64\pi\left[1 - \frac{2}{3}\right] = \boxed{\frac{64\pi}{3}}$$

$\rho \leq 3$

5. Let E be the region given by $y \geq 0$ and $z \geq 0$, and $x^2 + y^2 + z^2 \leq 9$..

a) (8 points) If you were integrating over E, would you integrate the region in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you choose the coordinate system you did, and express the bounds for the region E in that system.

spherical. region is upper hemisphere



$\rho \in [0, 3]$

$\varphi \in [0, \frac{\pi}{2}]$

$\theta \in [0, 2\pi]$

b) (12 points) Use integration to find the volume of E.

$$V = \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{3} 1 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_0^3 \rho^2 d\rho \cdot \int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi$$

$$= 2\pi \left[\frac{\rho^3}{3}\right]_0^3 \cdot \left[-\cos\varphi\right]_0^{\frac{\pi}{2}}$$

$$= 2\pi \cdot 9 \cdot -(0 - 1)$$

$$= \boxed{18\pi}$$

Check w/ geometry

$\frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \cdot 3^3 \cdot \frac{1}{2}$

$= 2\pi \cdot 9$ ✓

$= \boxed{18\pi}$

$= 2\pi \cdot 9$ ✓

6. Answer the following short answer questions.

a) (5 points) A continuous function $f(x,y)$ will have absolute extrema on a set D if D satisfies two conditions. What are those conditions?

closed & bounded

b) (5 points) If the average value of $f(x,y)$ is 5, and $\int\int_D f(x,y)dA = 30$, what is the area of a region D?

$$f_{avg} = \frac{\iint f\, dA}{\iint 1\, dA} \Rightarrow \text{area} = \frac{30}{5} = \boxed{6}$$

c) (5 points) Find the Jacobian of the transformation $x(u,v) = u^2 + v$, $y(u,v) = u + v^2$.

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & 1 \\ 1 & 2v \end{vmatrix} = \boxed{4uv - 1}$$