NAME Soln ______________________ Rec. Instructor: ______________

Signature ______________________ Rec. Time ______________

CALCULUS III - FINAL EXAM

<u>Show all work</u> for full credit. No books or notes are permitted.

Problem	Points	Possible	Problem	Points	Possible
1		9	6		20
2		20	7		15
3		10	8		20
4		20	9		16
5		20	EC		15
Total Score		79			71

Note: Bold letters, like $\mathbf{u}$, are considered vectors unless specified otherwise.

You are free to use the following formulas on any of the problems.

Projection: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$

Cylindrical Coordinates:

$$x = r\cos(\theta) \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin(\theta) \qquad \tan(\theta) = \frac{y}{x}$$

$$z = z \qquad z = z$$

$$dV = r\,dr\,d\theta\,dz$$

Spherical Coordinates:

$$x = \rho\cos(\theta)\sin(\varphi) \qquad \rho = \sqrt{x^2+y^2+z^2}$$

$$y = \rho\sin(\theta)\sin(\varphi) \qquad \tan(\theta) = \frac{y}{x}$$

$$z = \rho\cos(\varphi) \qquad \cos(\varphi) = \frac{z}{\rho}$$

$$dV = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

Second Derivative Test: Let $z = f(x,y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

Trig Identities: $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ $\quad\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Line Integrals:

$$\int_C f ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

Surface Integrals:

$$\int\int_S f dS = \int\int_R f(\mathbf{r}(u,v)) \|(\mathbf{t}_u \times \mathbf{t}_v)\| dudv$$

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v) dudv$$

Green's Theorem:

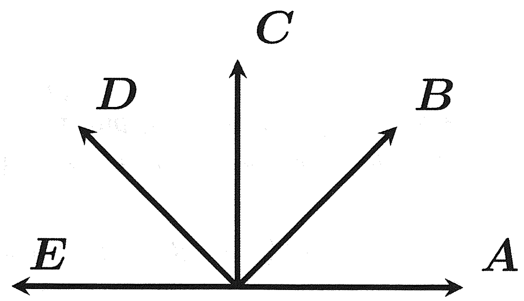
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \int\int_D (Q_x - P_y) dA$$

$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \int\int_D (P_x + Q_y) dA$$

Stokes' Theorem:

$$\int\int_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

1. (9 points) For this problem we refer to the following diagram, which is drawn to scale:



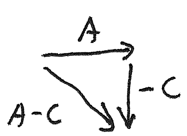
The vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$ all have length five. All of the angles between the vectors are multiples of 45 degrees. Compute the following explicitly:

a) $\mathbf{C} \cdot \mathbf{D}$
b) $\|\mathbf{B} \times \mathbf{E}\|$
d) $\|\mathbf{A} - \mathbf{C}\|$

a) $C \cdot D = \|C\| \, \|D\| \cos\theta = 25 \cos\frac{\pi}{4} = \boxed{\frac{25 \cdot \sqrt{2}}{2}}$

b) $\|B \times E\| = \|B\| \, \|E\| \sin\theta = 25 \sin\frac{3\pi}{4} = \boxed{\frac{25\sqrt{2}}{2}}$

c) $\|A - C\| = \boxed{5\sqrt{2}}$



2. (20 points) Use Lagrange Multipliers to find the maximum and minimum of the following function

$$f(x, y) = 2x + 6y$$

subject to the constraint

$$3x^2 + y^2 = 36.$$

$\nabla f = \lambda \nabla g$

$\langle 2, 6\rangle = \lambda \langle 6x, 2y\rangle$

$$\begin{cases} 2 = 6\lambda x \\ 6 = 2\lambda y \\ 3x^2 + y^2 = 36 \end{cases}$$

$x = \frac{1}{3\lambda}$, $y = \frac{3}{\lambda}$ $\Rightarrow y = 9x$

$3 \cdot \frac{1}{9\lambda^2} + \frac{9}{\lambda^2} = 36$

$\left(9 + \frac{1}{3}\right)\frac{1}{\lambda^2} = 36$

$3x^2 + 81x^2 = 36$

$84x^2 = 36$

$x^2 = \frac{36}{84} = \frac{18}{42} = \frac{9}{21} = \frac{3}{7}$

$x = \pm\sqrt{\frac{3}{7}}$

$y = \pm 9\sqrt{\frac{3}{7}}$

c.c.p.: $\left(\sqrt{\frac{3}{7}}, 9\sqrt{\frac{3}{7}}\right), \left(-\sqrt{\frac{3}{7}}, -9\sqrt{\frac{3}{7}}\right)$

$f\left(\sqrt{\frac{3}{7}}, 9\sqrt{\frac{3}{7}}\right) = \boxed{56\sqrt{\frac{3}{7}}}$ ← max $= 8\sqrt{21}$

$f\left(-\sqrt{\frac{3}{7}}, -9\sqrt{\frac{3}{7}}\right) = \boxed{-56\sqrt{\frac{3}{7}}}$ ← min $= -8\sqrt{21}$

3. (10 points) Let D be the region given by $[-1,1] \times [1,e]$. Compute the double integral

$$\int\int_D \frac{x}{y} dA.$$

$$= \int_1^e \int_{-1}^1 \frac{x}{y} \, dx \, dy$$

$$= \underbrace{\int_{-1}^1 x \, dx}_{=0} \cdot \int_1^e \frac{1}{y} \, dy$$

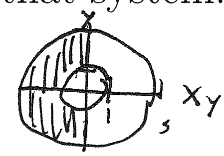
$$= \boxed{0}$$

r^2

4. Let E be the region such that $1 \le x^2 + y^2 \le 25$, $1 \le z \le 3$, and $y \le 0$.

a) (8 points) If you were evaluating the given integral over E, would you integrate in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you choose the coordinate system you did, and express the bounds for the region E in that system.

Cyl. Flat tops ($z=3$) Bottom



$$\boxed{\begin{array}{l} z \in [1,3] \\ r \in [1,5] \\ \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \end{array}}$$

b) (12 points) Use integration to find the volume of E.

$$vol(E) = \iiint_E 1\,dV = \int_1^3 dz \cdot \int_1^5 r\,dr \cdot \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta$$

Jacobian.

$$= 2 \cdot \left[\frac{r^2}{2}\right]_1^5 \cdot \pi$$

$$= (25-1)\pi = \boxed{24\pi}$$

P Q

5. Let $\mathbf{F}(x,y) = \langle ye^x + 6x - 1, e^x + 2y + 3\rangle$.

a) (10 points) Determine if $\mathbf{F}$ is conservative. If it is conservative, find a potential function. If it is not conservative, explain why it is not.

$$\left.\begin{array}{l} Q_x = e^x \\ P_y = e^x \end{array}\right) \Rightarrow \boxed{\text{conservative}}$$

$$\boxed{f = ye^x + 3x^2 - x + y^2 + 3y}$$

b) (10 points) Let C be the oriented curve with parameterization

$$\mathbf{r}(t) = \langle t^2 + t, 2 + t\rangle$$

for $0 \le t \le 1$. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

State any theorems used.

$r(0) = (0,2) = a$
$r(1) = (2,3) = b$

By FTLI, $\int_C \vec{F} \cdot d\vec{r} = f(r(1)) - f(r(0))$

$= f(2,3) - f(0,2)$

$= 3e^2 + 12 - 2 + 9 + 9 - (2 + 0 - 0 + 4 + 6)$

$= 3e^2 + 28 - 12$

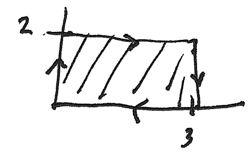
$= \boxed{3e^2 + 16}$

6. (20 points) Calculate the line integral

$$\oint_C \overset{P}{(y^2 - 2xy)}dx + \overset{Q}{(x^3 y)}dy$$

where C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,2)$, and $(0,2)$ oriented counterclockwise. State any theorems used.

By Green's thm,



$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

$$= \int_0^2\int_0^3 (3x^2y - 2y + 2x)\,dx\,dy$$

$$= \int_0^2 x^3y - 2xy + x^2\Big]_{x=0}^{3}\,dy$$

$$= \int_0^2 \underbrace{27y - 6y}_{=21y} + 9\,dy$$

$$= \frac{21y^2}{2} + 9y\Big]_0^2$$

$$= 42 + 18 = \boxed{\cancel{24}}\ \boxed{60}$$

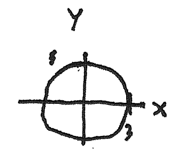
7. (15 points) Evaluate the integral

$$\int\int_S 1dS$$

where S is the portion of $x^2 + y^2 = 9$ between $z = 1$ and $z = 3$.

By integration,

$r(u,v) = (3\cos u, 3\sin u, v)$ $u \in [0, 2\pi]$, $v \in [1, 3]$



$r_u = \langle -3\sin u, 3\cos u, 0\rangle$

$r_v = \langle 0, 0, 1\rangle$

$r_u \times r_v = \langle 3\cos u, 3\sin u, 0\rangle$

$\|r_u \times r_v\| = 3$

Lateral surface of cylinder

$2\pi \cdot 3 \cdot (3-1)$

$= \boxed{12\pi}$

$$\iint_S 1\, dS = \iint_D 1 \cdot \|r_u \times r_v\|\, dA$$

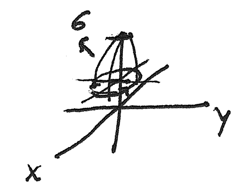
$$= \int_0^{2\pi}\int_1^3 3\, dA = 3 \cdot 2\pi \cdot 2 = \boxed{12\pi}$$

8. (20 points) Compute the integral

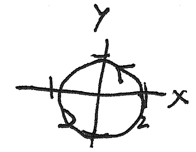
$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = x\mathbf{i} + (x^2 + y^2)\mathbf{j} + z^2\mathbf{k}$ and S is the paraboloid $z = 6 - x^2 - y^2$ where $z \geq 2$, oriented outward. State any theorems used.

Boundary of paraboloid at $z=2 \Rightarrow 4 = x^2+y^2$ is boundary



By Stokes thm,



$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} \qquad C = \partial S$$

$t \in [0, 2\pi]$

$r(t) = \langle 2\cos t, 2\sin t, 2\rangle$

$r'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} \langle 2\cos t, 4, 4\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} (-4\sin t \cos t + 8\cos t)\, dt$$

$$= -\frac{4}{2}\sin^2 t + 8\sin t \Big]_0^{2\pi}$$

$$= \boxed{0}$$

9. Answer the following short answer questions.

a) (2 points) Suppose that $\nabla f(1,0) = 0$, and the discriminant D of f at $(1,0)$ is positive. Do you need to know any more information to determine if $(1,0)$ is a local maximum?

Need that $f_{xx}(1,0) < 0$.

b) (5 points) Find the Jacobian of the transformation $x(u,v,w) = uvw$, $y(u,v,w) = u+v+w$, and $z(u,v,w) = u-v$.

$$J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} vw & uw & uv \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1.(uw-uv) - (-1)(vw-uv) + 0.$$

$$= uw - uv + vw - uv$$

$$= \boxed{uw - 2uv + vw}$$

c) (4 points) If f is a function of x and y, and x and y are each functions of r and s use the chain rule to express $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial s}$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

d) (5 points) Find the equation of the tangent plane to $f(x, y) = yx + x^2y^2$ at point $(2, 3)$.

$f(x,y) = yx + x^2y^2$ $\qquad$ $f(2,3) = 6 + 4\cdot 9 = 42$

$f_x = y + 2xy^2$ $\qquad$ $f_x(2,3) = 3 + 2\cdot 2\cdot 9 = 39$

$f_y = x + 2x^2y$ $\qquad$ $f_y(2,3) = 2 + 2\cdot 4\cdot 3 = 26$

$$\boxed{z = 42 + 39(x-2) + 26(y-3)}$$

EC. (15 points) Use the Divergence Theorem to evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = \langle y \sin(\pi x), y^2 z, z + xy \rangle$$

and S is the surface of the box with $-1 \le x \le 1$, $2 \le y \le 5$, and $0 \le z \le 1$. Note that all six sides of the box are included in S.

$$\operatorname{div} F = \pi y \cos(\pi x) + 2yz + 1$$

$$= \iiint_E \operatorname{div} \vec{F}\, dV$$

$$= \int_0^1 \int_2^5 \int_{-1}^1 \left(\pi y \cos(\pi x) + 2yz + 1\right) dx\, dy\, dz$$

$$= \int_0^1 \int_2^5 \left(\cancel{y \sin(\pi x)\Big]_{x=-1}^{1}}^{\,0} + 4yz + 2 \right) dy\, dz$$

$$= \int_0^1 \int_2^5 (4yz + 2)\, dy\, dz$$

$$= \int_0^1 2y^2 z + 2y \Big]_{y=2}^{5} dz$$

$$= \int_0^1 42z + 6\, dz$$

$$= 21z^2 + 6z \Big]_0^1$$

$$= 21 + 6 = \boxed{27}$$