U23_222_Practice_Test_3 live solve

Wednesday, July 12, 2023 9:55 AM

NAME Soln (Live Solve) Rec. Instructor: ____________

Signature ____________ Rec. Time ____________

CALCULUS III - PRACTICE TEST 3

Show all work for full credit. No books or notes are permitted.

Problem	Points	Possible
1		
2		
3		
4		
5		
6		
7		
Total Score		

Note: Bold letters, like **u**, are considered vectors unless specified otherwise.

You are free to use the following formulas on any of the problems.

Cylindrical Coordinates:

$$x = r\cos(\theta) \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin(\theta) \qquad \tan(\theta) = \frac{y}{x}$$
$$z = z \qquad z = z$$
$$dV = r\,dr\,d\theta\,dz$$

Spherical Coordinates:

$$x = \rho\cos(\theta)\sin(\varphi) \qquad \rho = \sqrt{x^2+y^2+z^2}$$
$$y = \rho\sin(\theta)\sin(\varphi) \qquad \tan(\theta) = \frac{y}{x}$$
$$z = \rho\cos(\varphi) \qquad \cos(\varphi) = \frac{z}{\rho}$$
$$dV = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

Second Derivative Test: Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

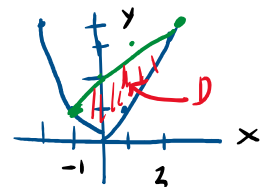
$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

Change of Variables: We have $T : S \to R$ and $T(u, v) = (g(u, v), h(u, v))$.
$\text{Jac}(T) = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|$ and

$$\int\int_S f(x, y)dxdy = \int\int_S f(g(u, v), h(u, v))|\text{Jac}(T)|dudv.$$

1. Integrate $f(x, y) = x$ over the region bounded by $y = x^2$ and $y = x + 2$.



$$x^2 = x+2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1$$

$$\iint_D f(x,y)\, dA$$

$$= \int_{-1}^{2}\int_{x^2}^{x+2} x \quad dy\, dx$$

$$= \int_{-1}^{2} x\, [y]_{y=x^2}^{x+2}\, dx$$

$$= \int_{-1}^{2} x\,(x + 2 - x^2)\, dx$$

$$= \int_{-1}^{2} (-x^3 + x^2 + 2x)\, dx$$

$$= \left. -\frac{x^4}{4} + \frac{x^3}{3} + x^2 \right]_{-1}^{2}$$

$$= -\cancel{4} + \frac{8}{3} + \cancel{4} - \left(-\frac{1}{4} - \frac{1}{3} + 1\right)$$

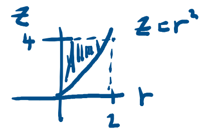
$$= \underbrace{\frac{8}{3} + \frac{1}{4} + \frac{1}{3}}_{=3} - 1 = 2 + \frac{1}{4} = \boxed{\frac{9}{4}}$$

$\theta \in [-\frac{\pi}{2}, 0]$

$0 \le r^2 \le z \le 4$

2. Let R be the region given by $x \geq 0$, $y \leq 0$, and $x^2 + y^2 \leq z \leq 4$.

a) If you were integrating over R, would you integrate the region in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you choose the coordinate system you did, and express the bounds for the region R in that system.



Cyl.

- 3D (z)
- flat top ⇝ cyl better spherical

$r \in [0, 2]$

$z \in [r^2, 4]$

$$R = \{(r, \theta, z) \mid -\frac{\pi}{2} \le \theta \le 0,\ 0 \le r \le 2,\ r^2 \le z \le 4\}$$

b) Find the volume of R.

$$\text{vol}(R) = \iiint_R 1 \, dV$$

$$= \int_{-\frac{\pi}{2}}^{0} \int_0^2 \int_{r^2}^4 1 \cdot r \, dz \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0} 1 \, d\theta \cdot \int_0^2 \int_{r^2}^4 r \, dz \, dr$$

$$= \frac{\pi}{2} \cdot \int_0^2 r \underbrace{[z]_{r^2}^4}_{= 4 - r^2} dr$$

$$\int_0^2 4r - r^3 \, dr$$

$$= \frac{\pi}{2} \cdot \left[2r^2 - \frac{r^4}{4}\right]_0^2 = \frac{\pi}{2}(8 - 4)$$

$$= \boxed{2\pi}$$

3. Find and classify all the critical points of $3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

f

$$\nabla f = \langle 6xy - 6x,\ 3x^2 + 3y^2 - 6y \rangle = \vec{0}$$

$$\Rightarrow \begin{cases} 6x(y-1) = 0 & \Rightarrow x=0 \text{ or } y=1 \\ 3x^2 + 3y^2 - 6y = 0 \end{cases}$$

If $x=0$, $3y^2 - 6y = 0 \Rightarrow 3y(y-2) = 0 \Rightarrow y = 0, 2$

$(0,0)$, $(0,2)$

If $y=1$, $3x^2 + 3 - 6 = 0$

$\Rightarrow 3x^2 - 3 = 0$

$\Rightarrow 3(x^2-1) = 0 \Rightarrow x = \pm 1$

$(1,1)$, $(-1,1)$

$f_{xx} = 6y - 6 = 6(y-1)$

$f_{yy} = 6y - 6 = 6(y-1)$

$f_{xy} = 6x$

$\rightarrow D = f_{xx} f_{yy} - f_{xy}^2$

$= 36(y-1)^2 - 36x^2$

$D(0,0) = 36 > 0$, $f_{xx}(0,0) = -6 < 0$

$D(0,2) = 36 > 0$, $f_{xx}(0,2) = 6 > 0$

$D(1,1) = 0 - 36 < 0$

$D(-1,1) = 0 - 36 < 0$

local max @ $(0,0)$

local min @ $(0,2)$

saddle @ $(1,1)$

saddle @ $(-1,1)$

4. Use Lagrange Multipliers to find the maximum and minimum of the following function

$$f(x, y, z) = x + y^2 - z$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 = 1.$$

$g = x^2 + y^2 + z^2 - 1 = 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} \langle 1, 2y, -1 \rangle = \lambda \langle 2x, 2y, 2z \rangle \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 1 = 2\lambda x \\ 2y = 2\lambda y \Rightarrow 2\lambda y - 2y = 0 \Rightarrow 2y(\lambda - 1) = 0 \\ -1 = 2\lambda z \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

If $\lambda = 1$:

$1 = 2x \Rightarrow x = \frac{1}{2}$

$-1 = 2z \Rightarrow z = -\frac{1}{2}$

$\Rightarrow \frac{1}{4} + y^2 + \frac{1}{4} = 1$

$\Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm \frac{1}{\sqrt{2}}$

$f = x + y^2 - z$

$\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) \xrightarrow{f} \frac{1}{2} + \frac{1}{2} - \left(-\frac{1}{2}\right)$

$= \frac{3}{2} = 1.5$

If $y = 0$:

(Assume $\lambda \neq 0$)

$$\begin{cases} 1 = 2\lambda x & \Rightarrow \frac{1}{2\lambda} = x \\ -1 = 2\lambda z & \Rightarrow -\frac{1}{2\lambda} = z = -x \\ x^2 + z^2 = 1 \end{cases}$$

$\Rightarrow x^2 + (-x)^2 = 1$

$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

$\left(\pm\frac{1}{\sqrt{2}}, 0, \mp\frac{1}{\sqrt{2}}\right) \leftarrow$ 2 points

$f \hookrightarrow \frac{1}{\sqrt{2}} + 0 - \left(-\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \approx 1.4$

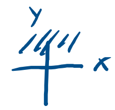
$-\frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} \approx -1.4$

$$\boxed{\begin{array}{l} \text{Max: } \frac{3}{2} \\ \text{Min: } -\sqrt{2} \end{array}}$$

$\rho^2 = 25$

5. Let E be the region inside $x^2 + y^2 + z^2 = 25$ and $z = -\sqrt{3x^2 + 3y^2}$, and where $y \geq 0$.

a) If you were integrating over E, would you integrate the region in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you choose the coordinate system you did, and express the bounds for the region E in that system.



$\theta \in [0, \pi]$

Sph.

– Cyl looks bad.

$z = -\sqrt{3}\sqrt{x^2+y^2}$

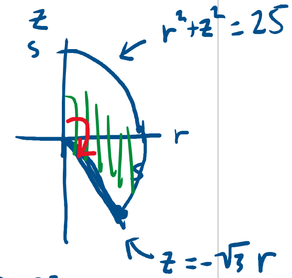
$z = -\sqrt{3}\, r$

$\rho\cos\varphi = -\sqrt{3}\,\rho\sin\varphi$

$\frac{-1}{\sqrt{3}} = \tan\varphi$

$\varphi = \frac{5\pi}{6}$

$\varphi \in [0, \frac{5\pi}{6}]$ $\qquad$ $\rho \in [0, 5]$



$\frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

$$\boxed{E = \{(\rho, \varphi, \theta) \mid 0 \le \rho \le 5,\ 0 \le \varphi \le \tfrac{5\pi}{6},\ 0 \le \theta \le \pi\}}$$

b) Evaluate

$$\iiint_E x\,dV.$$

$x = \rho\cos(\theta)\sin(\varphi)$

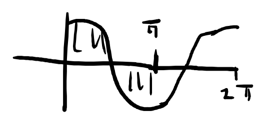
$y = \rho\sin(\theta)\sin(\varphi)$

$z = \rho\cos(\varphi)$

$dV = \rho^2\sin(\phi)d\rho d\theta d\phi$

$$= \int_0^{\pi}\int_0^{\frac{5\pi}{6}}\int_0^{5} \rho\cos\theta\sin\varphi \cdot \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^5 \rho^3\, d\rho \cdot \underline{\int_0^{\frac{5\pi}{6}} \sin^2\varphi\, d\varphi} \cdot \underbrace{\int_0^{\pi} \cos\theta\, d\theta}_{=0}$$



$\sin(\pi) = 0$

$\sin(0) = 0$

$= \boxed{0}$

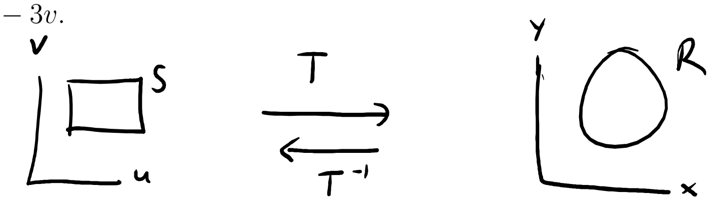
$\cos 2x = 2\cos^2 x - 1 \Rightarrow \frac{1}{2}(\cos 2x + 1) = \cos^2 x$

$= 1 - 2\sin^2 x \Rightarrow \frac{1}{2}(1 - \cos 2x) = \sin^2 x$

6. Evaluate

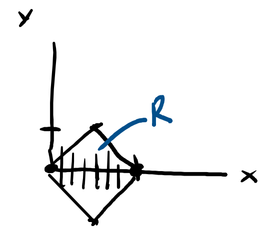
$$\int\int_R x + y dA$$

where R is the trapezoidal region with vertices given by the points $(0,0)$, $(2,0)$, $(1,1)$ and $(1,-1)$ using the transformation $x(u,v) = 2u+3v$ and $y(u,v) = 2u - 3v$.



$$T : \begin{cases} x = 2u+3v \\ y = 2u-3v \end{cases}$$

$$\left.\begin{aligned} x+y = 4u &\Rightarrow u = \tfrac{1}{4}(x+y) \\ x-y = 6v &\Rightarrow v = \tfrac{1}{6}(x-y) \end{aligned}\right\} T^{-1}$$



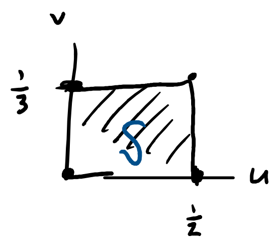
$T^{-1}(x,y) = (u,v)$

$T^{-1}(0,0) = (0,0)$

$T^{-1}(2,0) = (\frac{1}{2}, \frac{1}{3})$

$T^{-1}(1,1) = (\frac{1}{2}, 0)$

$T^{-1}(1,-1) = (0, \frac{1}{3})$



$$J(T) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6-6 = -12$$

$$\iint_R (x+y)\, dA = \iint_S 4u\, |J(T)|\, du\, dv$$

$$= \int_0^{\frac{1}{3}}\int_0^{\frac{1}{2}} 4u \cdot \underline{12}\, du\, dv$$

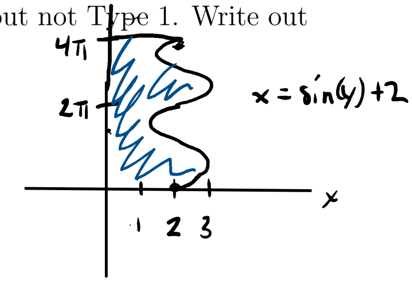
$$= 48 \int_0^{\frac{1}{2}} u\, du \cdot \int_0^{\frac{1}{3}} 1\, dv = 48 \left[\frac{u^2}{2}\right]_0^{\frac{1}{2}} \cdot \frac{1}{3}$$

$$= 48 \cdot \frac{1}{8} \cdot \frac{1}{3} = \boxed{2}$$

7. Answer the following short answer questions.

a) Give an example of a region that is Type 2, but not Type 1. Write out the region explicitly, don't just graph it.

$$\boxed{R = \{(x,y) \mid 0 \le y \le 4\pi,\ 0 \le x \le \sin(y)+2\}}$$



b) Write out the Extreme Value Theorem (Theorem 4.18 from the textbook).

If $f(x,y)$ is continuous on a closed, bounded set D,

Then f attains its min and max values on D.

c) If the ~~area~~ volume of a region E is 2, and $\int\int\int_E f(x,y,z)dV = 3$, what is the average value of $f(x,y,z)$?.

$$f_{avg} = \frac{\iiint_E f\,dV}{\iiint_E 1\,dV} = \boxed{\frac{3}{2}}$$

d) If I set up an integral that looks like:

$$\int_0^1 \int_0^z \int_0^{x+y} f(x,y,z)dzdydx$$

what is the problem with my integral set up?

y-integral depends on z

and z-integral depends on y

cyclical dependencies not allowed

Region being integrated over is nonsensical

e) Suppose we want to integrate

$$\int\int_R y\sin(xy)dA$$

over $R = [1,2] \times [0,\pi]$. Which is less steps, integrating with respect to x first, or y first? Why?

$\boxed{\text{Integrating x First is easier } (dx\,dy)}$

The y gets absorbed:

$$\int_1^2 y\sin(xy)\,dx = -\cos(xy)\Big]_{x=1}^{2}$$

Trying y-first will require I.B.P.

$$\int_0^\pi y\sin(xy)dy$$