Practice Final Sol

Tuesday, July 25, 2023 1:21 PM

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NAME Solns

Rec. Instructor: ____________

Signature ____________

Rec. Time ____________

CALCULUS III - PRACTICE TEST FINAL

Show all work for full credit. No books or notes are permitted.

Problem	Points	Possible	Problem	Points	Possible
1			7		
2			8		
3			9		
4			10		
5			11		
6			EC		
Total Score					

Note: Bold letters, like $\mathbf{u}$, are considered vectors unless specified otherwise.

You are free to use the following formulas on any of the problems.

Projection: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$

Cylindrical Coordinates:

$$x = r\cos(\theta) \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin(\theta) \qquad \tan(\theta) = \frac{y}{x}$$
$$z = z \qquad z = z$$
$$dV = r\,dr\,d\theta\,dz$$

Spherical Coordinates:

$$x = \rho\cos(\theta)\sin(\varphi) \qquad \rho = \sqrt{x^2+y^2+z^2}$$
$$y = \rho\sin(\theta)\sin(\varphi) \qquad \tan(\theta) = \frac{y}{x}$$
$$z = \rho\cos(\varphi) \qquad \cos(\varphi) = \frac{z}{\rho}$$
$$dV = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

1

Second Derivative Test: Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

Trig Identities: $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ $\qquad \sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Line Integrals:

$$\int_C f ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$$

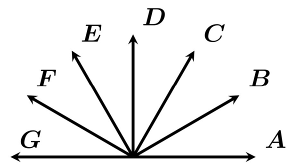
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

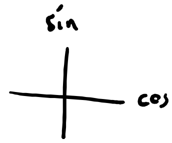
Surface Integrals:

$$\int\int_S f dS = \int\int_R f(\mathbf{r}(u, v))\|(\mathbf{t}_u \times \mathbf{t}_v)\|dudv$$

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)dudv$$

1. For this problem we refer to the following diagram, which is drawn to scale:





The vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$, and $\boldsymbol{G}$ all have length three. All of the angles between the vectors are multiples of 30 degrees. Compute the following explicitly:

a) $\mathbf{A}\cdot\mathbf{E}$
b) $\|\mathbf{B}\times\mathbf{F}\|$
c) $\mathbf{C}\cdot\mathbf{E}$
d) $\|\mathbf{D}-\mathbf{G}\|$
e) $\mathbf{A}\cdot\mathbf{A}$

a) $A\cdot E = \|A\|\cdot\|E\|\cos\theta$

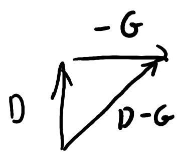
$= 9\cos\frac{2\pi}{3} = \boxed{-\frac{9}{2}}$

b) $\|B\times F\| = \|B\|\,\|F\|\sin\theta$

$= 9\sin\frac{2\pi}{3} = \boxed{\frac{9\sqrt{3}}{2}}$

c) $C\cdot E = \|C\|\,\|E\|\cos\theta$

$= 9\cos\frac{\pi}{3} = \boxed{\frac{9}{2}}$



d) $\|D-G\| = \boxed{3\sqrt{2}}$

e) $A\cdot A = \|A\|^2 = \boxed{9}$

2. Use Lagrange Multipliers to find the maximum and minimum of the following function

$$f(x, y, z) = x^2 + 3z^2$$

subject to the constraint

$$x^2 + y^2 + 4z^2 = 36.$$

$g = x^2 + y^2 + 4z^2 - 36$

$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

$\Rightarrow \begin{cases} \langle 2x, 0, 6z \rangle = \lambda \langle 2x, 2y, 8z \rangle \\ g = 0 \end{cases}$

$\Rightarrow \begin{cases} 2x = 2\lambda x \\ 0 = 2\lambda y \\ 6z = 8\lambda z \\ x^2 + y^2 + 4z^2 = 36 \end{cases}$

Eq 2: $\lambda = 0$ or $y = 0$

If $\lambda = 0$,

Eqns 1,3 $\Rightarrow$ $x = 0$, $z = 0$

Constraint $\Rightarrow$ $y^2 = 36$

$\Rightarrow y = \pm 6$

$\Rightarrow \underline{(0, \pm 6, 0)}$

If $y = 0$, system reduces to

$\begin{cases} 2x = 2\lambda x \\ 6z = 8\lambda z \\ x^2 + 4z^2 = 36 \end{cases} \Rightarrow \begin{cases} 2x(1-\lambda) = 0 \\ 2z(3 - 4\lambda) = 0 \\ x^2 + 4z^2 = 36 \end{cases}$

Eqn 1 $\Rightarrow$ $x = 0$ or $\lambda = 1$

If $x = 0$, constraint $\Rightarrow 4z^2 = 36 \Rightarrow z = \pm 3 \Rightarrow \underline{(0, 0, \pm 3)}$

If $\lambda = 1$, eqn 2 $\Rightarrow 6z = 8z \Rightarrow z = 0$

constraint $\Rightarrow x^2 = 36 \Rightarrow x = \pm 6 \Rightarrow \underline{(\pm 6, 0, 0)}$

$f(\pm 6, 0, 0) = \boxed{36 \leftarrow \text{max}}$

$f(0, \pm 6, 0) = \boxed{0 \leftarrow \text{min}}$

$f(0, 0, \pm 3) = 27$

$x \qquad y \qquad z$

3. Let E be the region given by $[0,1] \times [0,2] \times [0,3]$. Compute the triple integral

$$\int\int\int_E z + xe^{xy} dV.$$

$$= \int_0^3\int_0^2\int_0^1 z \, dx\,dy\,dz + \int_0^3\int_0^1\int_0^2 xe^{xy} \, dy\,dx\,dz$$

$$= 2 \cdot \int_0^3 z\,dz + 3\int_0^1\int_0^2 xe^{xy}\,dy\,dx$$

$$= 2 \cdot \frac{z^2}{2}\Big]_0^3 + 3\int_0^1 \left[e^{xy}\right]_{y=0}^2 dx$$

$$= 9 + 3\int_0^1 (e^{2x} - 1)\,dx$$

$$= 9 + 3\left[\frac{e^{2x}}{2} - x\right]_0^1$$

$$= \boxed{9 + 3\left[\frac{e^2}{2} - 1 - \left(\frac{1}{2}\right)\right]}$$

$$= 9 + \frac{-9}{2} + \frac{3e^2}{2} = \boxed{\frac{9}{2} + \frac{3e^2}{2}}$$

4. Calculate the following line integrals:

a) $\int_C f\,ds$ where $f(x,y,z) = 3x - yz$ and C is the line between the points $(1,3,0)$ and $(2,5,4)$.

$$\int_C f\,ds = \int_a^b f(\vec{r}(t)) \cdot \|\vec{r}\,'(t)\|\,dt$$

$\vec{r}(t) = \langle 1+t, 3+2t, 4t \rangle$, $t \in [0,1]$

$r' = \langle 1, 2, 4 \rangle$

$\|r'\| = \sqrt{1+4+16} = \sqrt{21}$

$$\int_C f\,ds = \int_0^1 [3(1+t) - (3+2t)4t]\sqrt{21}\,dt$$

$$= \sqrt{21}\int_0^1 (3 - 9t - 8t^2)\,dt$$

$$= \sqrt{21}\left[3t - \frac{9}{2}t^2 - \frac{8}{3}t^3\right]_0^1 = \boxed{\sqrt{21} \cdot -\frac{25}{6}}$$

b) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle 6y, 2 \rangle$ and C is the piece of the graph $y = x^2$ between $x = 0$ and $x = 3$.

$\vec{r}(t) = \langle t, t^2 \rangle \quad t \in [0,3]$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

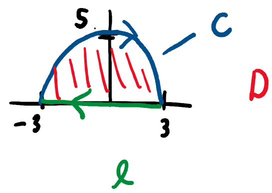
$$= \int_0^3 \langle 6t^2, 2 \rangle \cdot \langle 1, 2t \rangle\,dt$$

$$= \int_0^3 (6t^2 + 4t)\,dt$$

$$= 2t^3 + 2t^2 \Big]_0^3$$

$$= 54 + 18 = \boxed{72}$$

5. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = (y^2-4y+5)\mathbf{i}+(2xy-4x+9)\mathbf{j}$ on the upper half of the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$, oriented clockwise. State any theorems used.



$\vec{r}(t) = \langle 3\cos t, 5\sin t\rangle \quad 0 \to \pi$

oriented ccw

$\vec{r}(t) = \langle -3\cos t, 5\sin t\rangle \quad 0 \to \pi$

oriented c.w.

Green's thm

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA$$

$F = \langle P, Q\rangle$

$Q_x = 2y - 4$

$P_y = 2y - 4$

$$-\oint_{C\cup l} \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA = \iint_D 0\, dA = 0$$

$\vec{r}_l(t) = \langle t, 0\rangle \quad t: 3 \to -3$

$r_l' = \langle 1, 0\rangle$

$$\int_l \vec{F}\cdot d\vec{r} = \int_3^{-3} \vec{F}(\vec{r}_l(t))\cdot \vec{r}_l{}'(t)\, dt$$

$$= \int_3^{-3} \langle 5, -4t+9\rangle \cdot \langle 1, 0\rangle\, dt$$

$$= -\int_{-3}^{3} 5\, dt = -5\cdot 6 = -30$$

$$\int_C = \int_{C\cup l} - \int_l$$

$$= 0 - (-30)$$

$$\int_C = \boxed{30}$$

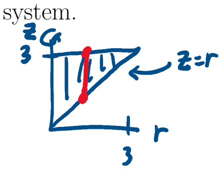
$0 \le r \le z \le 3$

6. Let E be the region such that $\sqrt{x^2+y^2} \le z \le 3$ and $x \ge 0$. Consider

$$\iiint_E x\,dV.$$

a) If you were evaluating the given integral over E, would you integrate in Cartesian, Polar, Cylindrical, or Spherical Coordinates? Explain why you choose the coordinate system you did, and express the bounds for the region E in that system.

Cyl. cone w/ flat top



$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$r \in [0,3]$

$z \in [r, 3]$

b) Evaluate

$$\iiint_E x\,dV.$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \int_r^3 r\cos\theta \cdot r \, dz\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \cdot \int_0^3 \int_r^3 r^2\, dz\, dr$$

$$= \sin\theta \Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \int_0^3 r^2(3-r)\, dr$$

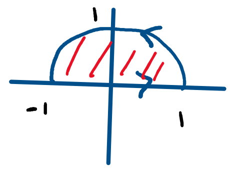
$$= [1-(-1)] \cdot \left[r^3 - \frac{r^4}{4}\right]_0^3 = 2\cdot\left(27 - \frac{81}{4}\right) = \boxed{\frac{27}{2}}$$

7. Calculate the line integral

$$\oint_C \overbrace{(6y+x^2)}^{P}dx + \overbrace{(1-2xy)}^{Q}dy$$

where C is the boundary of the upper half of the unit circle (this includes the x-axis), oriented counterclockwise. State any theorems used.

Green's thm



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA$$

$$= \iint_D (-2y-6)\,dA$$

alt.

$$-2\iint_D (y+3)\,dA$$

$$= -2\int_0^{\pi}\int_0^1 (r\sin\theta+3)\,r\,dr\,d\theta$$

$$\vdots$$

$$= -\frac{4}{3} - 3\pi$$

$$= -2\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (y+3)\,dy\,dx$$

$$= -2\int_{-1}^{1} \underbrace{\tfrac{1}{2}(1-x^2)}_{\text{even}} + 3\sqrt{1-x^2}\,dx$$

$$= -2\int_0^1 (1-x^2)\,dx + -6\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

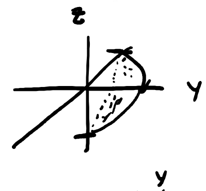
$$= -2\left[x-\frac{x^3}{3}\right]_0^1 - 6\cdot\frac{1}{2}\pi\cdot 1^2$$

$$= \boxed{-\frac{4}{3} - 3\pi}$$

8. Evaluate the integral

$$\int\int_S xz dS$$

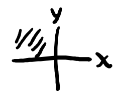
where S is the portion of the sphere of radius 1 where $x \le 0$, $y \ge 0$, $z \le 0$.



$\rho = 1$

$u = \theta \in [\frac{\pi}{2}, \pi]$

$v = \varphi \in [\frac{\pi}{2}, \pi]$



$\vec{r}(u,v) = \langle \cos u \sin v,\ \sin u \sin v,\ \cos v \rangle$

$\vec{t}_u = \vec{r}_u = \langle -\sin u \sin v, \cos u \sin v, 0 \rangle$

$\vec{t}_v = \vec{r}_v = \langle \cos u \cos v,\ \sin u \cos v,\ -\sin v \rangle$

$\vec{t}_u \times \vec{t}_v = \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v \rangle$

$= \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v \rangle$

$\|\vec{t}_u \times \vec{t}_v\| = \sqrt{\sin^4 v \cos^2 u + \sin^4 v \sin^2 u + \sin^2 v \cos^2 v}$

$= \sqrt{\sin^4 v + \sin^2 v \cos^2 v}$

$= \sqrt{\sin^2 v}$ $\quad v \in [\frac{\pi}{2}, \pi]$

$= \sin v$ $\quad \sin v > 0$

$$\iint_S f\, dS = \iint_S f(\vec{r}(u,v))\, \|\vec{t}_u \times \vec{t}_v\|\, du\, dv$$

$$\iint_S xz\, dS = \int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos u \sin v \cos v \cdot \sin v\, du\, dv$$

$$= \int_{\frac{\pi}{2}}^{\pi} \cos u\, du \cdot \int_{\frac{\pi}{2}}^{\pi} \sin^2 v \cos v\, dv$$

$$= \sin u \Big]_{\frac{\pi}{2}}^{\pi} \cdot \left[\frac{1}{3}\sin^3 v\right]_{\frac{\pi}{2}}^{\pi}$$

$$= (0-1)\cdot \frac{1}{3}(0-1)$$

$$= \boxed{\frac{1}{3}}$$

9. Compute the integral

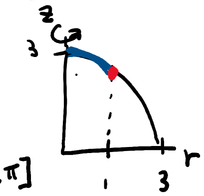
Stokes' thm

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \vec{F} \cdot d\vec{r}$$

where $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies in the cylinder $x^2 + y^2 = 1$ and above the xy-plane. State any theorems used.

$1 + z^2 = 9 \Rightarrow z = \pm 2\sqrt{2}$

$= 2\sqrt{2}$



$\vec{r}(t) = \langle \cos t, \sin t, 2\sqrt{2} \rangle \quad t \in [0, 2\pi]$

$$\oint_C \vec{F} \cdot dr = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle 2\sqrt{2}\cos t, 2\sqrt{2}\sin t, 2\sqrt{2}\cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} 0 \, dt = \boxed{0}$$

10. For the following, determine if the vector field is conservative or not. If it is, find a potential function.

a) $\mathbf{F}(x, y, z) = \langle \frac{2}{3}y^3z^2, 2xy^2z^2, x^2y^2z^2 \rangle$.

P Q R

$Q_x = 2y^2z^2$ $R_x = 2xy^2z^2$

$P_y = 2y^2z^2$ $P_z = \frac{4}{3}y^3z$

$R_x \neq P_z \Rightarrow \boxed{\vec{F} \text{ not conservative}}$

b) $\mathbf{F}(x, y) = \langle 6x^2 - 2xy^2 + \frac{y}{2\sqrt{x}}, -2x^2y + 4 + \sqrt{x} \rangle$.

P Q

$P_y = -4xy + \frac{1}{2\sqrt{x}}$

$Q_x = -4xy + \frac{1}{2\sqrt{x}}$

$\boxed{\text{Conservative}}$

$f = \int P\,dx = 2x^3 - x^2y^2 + y\sqrt{x} + C(y)$ $\frac{y}{2}\int x^{-\frac{1}{2}}\,dy$

$f = \int Q\,dy = -x^2y^2 + 4y + y\sqrt{x} + \tilde{C}(x)$ $y\,x^{\frac{1}{2}}$

$$\boxed{f = 2x^3 - x^2y^2 + 4y + y\sqrt{x} + C}$$

11. Answer the following short answer questions.

a) Suppose that $\nabla f(1,1) = 0$, and $f_{xx}(1,1)$ and $f_{yy}(1,1)$ are both positive. Do you need to know any more information to determine if $(1,1)$ is a local minimum?

$\hookrightarrow D(1,1) > 0, \quad f_{xx}(1,1) > 0$

$D = f_{xx} f_{yy} - f_{xy}^2$

$\boxed{\text{Need that } D(1,1) > 0}$

b) Suppose that $\nabla g(0,0) = 0$, and and the discriminant D of g is 0 at $(0,0)$. What does the Second Derivative Test tell us about how g behaves at $(0,0)$?

$D = 0 \Rightarrow$ Second Derivative Test $\boxed{\text{inconclusive}}$

c) If I set up an integral that looks like:

$$\int_0^z \int_2^{-1} \int_0^{y+3} f(x,y,z)\,dz\,dx\,dy$$

what is the problem with my integral set up?

z-integral depends on y

y-integral depends on z ← reaching inside, bad

$\boxed{\text{cyclic dependancy not allowed}}$

d) Find the Jacobian of the transformation $x(\rho, \theta, \varphi) = \rho\cos(\theta)\sin(\varphi)$, $y(\rho, \theta, \varphi) = \rho\sin(\theta)\sin(\varphi)$, and $z(\rho, \theta, \varphi) = \rho\cos(\varphi)$.

$$J = \begin{vmatrix} x_\rho & x_\theta & x_\varphi \\ y_\rho & y_\theta & y_\varphi \\ z_\rho & z_\theta & z_\varphi \end{vmatrix} = \boxed{\rho^2 \sin\varphi}$$

e) Find the equation of the tangent plane to $f(x, y) = y\cos(x) + \sin(xy)$ at point $\left(\frac{\pi}{4}, 2\right)$.

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$f(x,y) = y\cos x + \sin(xy)$ $\quad f(\frac{\pi}{4}, 2) = 2\frac{\sqrt{2}}{2} + \sin(\frac{\pi}{2})$

$= \sqrt{2} + 1$

$f_x = -y\sin x + y\cos(xy)$ $\quad f_x(\frac{\pi}{4}, 2) = -2\frac{\sqrt{2}}{2} + 2\cancel{\cos\frac{\pi}{2}}^{0}$

$= -\sqrt{2}$

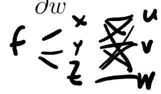
$f_y = \cos x + x\cos(xy)$ $\quad f_y(\frac{\pi}{4}, 2) = \frac{\sqrt{2}}{2} + \frac{\pi}{4}\cancel{\cos\frac{\pi}{2}}^{0}$

$= \frac{\sqrt{2}}{2}$

$$\boxed{z = \sqrt{2} + 1 + (-\sqrt{2})(x - \tfrac{\pi}{4}) + \tfrac{\sqrt{2}}{2}(y - 2)}$$

f) If f is a function of x, y, z, and x, y, and z are each functions of u, v and w, use the chain rule to express $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial w}$.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$



$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial w}$$

g) Compute the divergence and curl of $\mathbf{F} = \langle x^2y, xyz, -x^2z^2\rangle$.

$$\text{div } \vec{F} = \boxed{2xy + xz + -2x^2z}$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2y & xyz & -x^2z^2 \end{vmatrix} = \langle -xy, -(-2xz^2), yz - x^2\rangle$$

$$= \boxed{\langle -xy, 2xz^2, yz - x^2\rangle}$$

h) Calculate the limit if it exists. If the limit does not exist, explain why not.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4}$$

Along path $x = y^2$, $\displaystyle\lim_{y\to 0} \frac{y^4}{4y^4} = \frac{1}{4}$

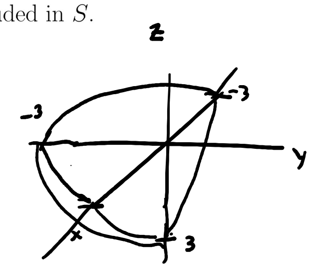
Along x-axis $(y=0)$, $\displaystyle\lim_{x\to 0} \frac{0}{x^2+0} = 0$

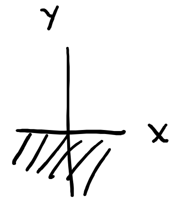
$\therefore$ $\boxed{\text{Limit DNE, since different paths gave different values.}}$

EC. Use the Divergence Theorem to evaluate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$ where

$$\mathbf{F}(x,y,z) = \langle x^2, y+z, xy\rangle$$

and S is the sphere of radius 3 with $z \le 0$ and $y \le 0$. Note that all three surfaces of this solid are included in S.





By Divergence Thm,

$$\iint_S \vec{F}\cdot dS = \iiint_E \operatorname{div}\vec{F}\, dV$$

$$= \int_{-\pi}^{0}\int_{\frac{\pi}{2}}^{\pi}\int_0^3 (2\rho\cos\theta\sin\varphi + 1)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$\rho \in [0,3]$

$\theta \in [-\pi, 0]$

$\varphi \in [\frac{\pi}{2}, \pi]$

$\operatorname{div} F = 2x + 1$

$$= \iiint (2\rho^3\cos\theta\sin^2\varphi + \rho^2\sin\varphi)\, dV$$

$$= 2\int_0^3 \rho^3\, d\rho \cdot \int_{\frac{\pi}{2}}^{\pi}\sin^2\varphi\, d\varphi \cdot \underbrace{\int_{-\pi}^{0}\cos\theta\, d\theta}_{=0} + \int_0^3 \rho^2\, d\rho \cdot \int_{\frac{\pi}{2}}^{\pi}\sin\varphi\, d\varphi \cdot \int_{-\pi}^{0} 1\, d\theta$$

$$= 0 + \pi\left[\frac{\rho^3}{3}\right]_0^3 \cdot \left[-\cos\varphi\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\pi \cdot 9\,(-1-0)$$

$$= \boxed{9\pi}$$