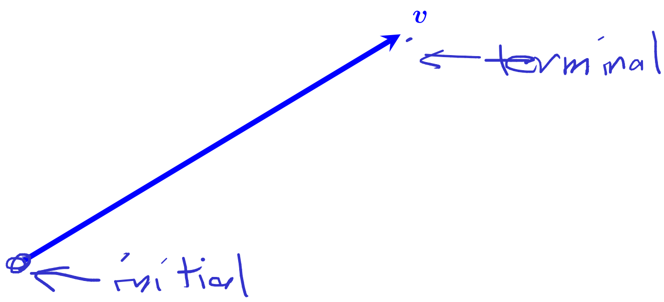
June 5: §2.1: Vectors in the Plane

- What is a vector?
  - A vector is a quantity that consists of two things:
    A direction and a magnitude (length).
  - A vector in the plane is represented by a directed line segment (an arrow).



$\vec{v}$

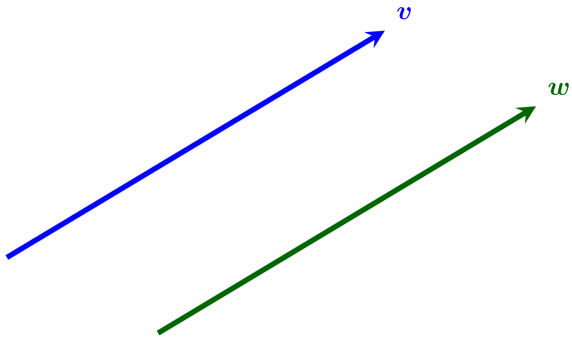
  - In print, it is common to denote a vector using boldface. When writing a vector by hand, it is easier to draw an arrow over the variable (e.g. $\vec{v}$ vs. $\mathbf{v}$)
  - The endpoints of the segment are called the **initial point** and the **terminal point** of the vector.

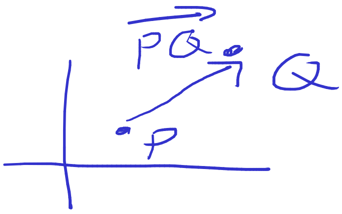
$\|\vec{v}\|$

  - An arrow from the initial point to the terminal point indicates the direction of the vector.
  - The length of the line segment represents its **magnitude**.
    * The notation $\|\mathbf{v}\|$ is used to denote the magnitude of the vector $\mathbf{v}$.
- Zero vector

$\vec{0}$

  - A vector whose initial point and terminal point are the same is called the **zero vector**, denoted $\mathbf{0}$.
  - The zero vector is the only vector without a direction. By convention, it can be considered to have any direction convenient to the problem at hand.
- Equivalent vectors
  - Vectors are **equivalent** if they have the same magnitude and direction.





We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\mathbf{v}$ and $\mathbf{w}$ are equivalent, we write

$$\mathbf{v} = \mathbf{w}$$

- When a vector has initial point P and terminal point Q the notation $\overrightarrow{PQ}$ is used.
  - Alert! Pay attention to the ordering of the letters. Starts at P, goes to Q. The vector $\overrightarrow{QP}$ goes in the opposite direction.
- Operations on vectors
  - Scalar multiplication
    * A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars have magnitude only.
    * Multiplying a vector by a scalar changes the vector's magnitude. This is called **scalar multiplication** and is denoted $k\mathbf{v}$, where $k \in \mathbb{R}$.
      · For $k < 0$, the direction is reversed.
      · The case $k = -1$ is denoted usually denoted $-\mathbf{v}$ instead of $(-1)\mathbf{v}$

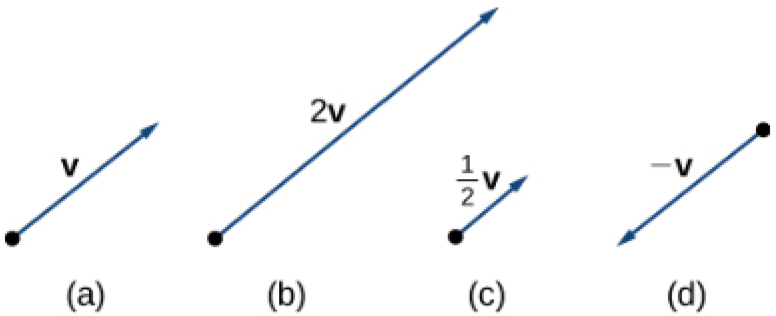
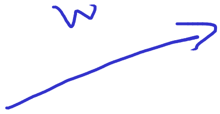


Figure 2.4 (a) The original vector $\mathbf{v}$ has length n units. (b) The length of $2\mathbf{v}$ equals $2n$ units. (c) The length of $\mathbf{v}/2$ is $n/2$ units. (d) The vectors $\mathbf{v}$ and $-\mathbf{v}$ have the same length but opposite directions.

  - Vector Addition
    * Given two vectors $\mathbf{v}, \mathbf{w}$, their sum is denoted $\mathbf{v} + \mathbf{w}$
    * Vector addition can be visualize in two different ways (the book calls these the **triangle method** and the **parallelogram method**):

$\vec{w}$



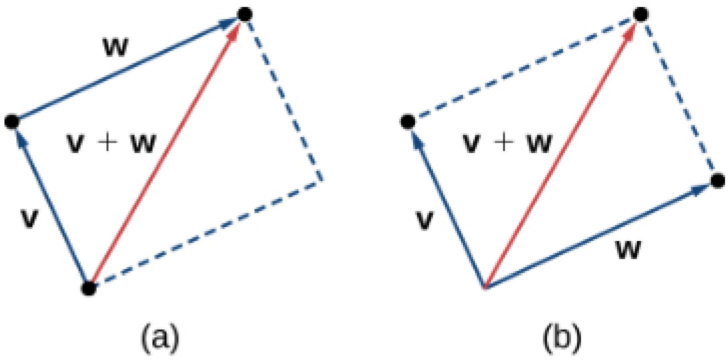
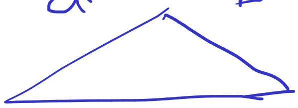


Figure 2.5 (a) When adding vectors by the triangle method, the initial point of $\mathbf{w}$ is the terminal point of $\mathbf{v}$. (b) When adding vectors by the parallelogram method, the vectors $\mathbf{v}$ and $\mathbf{w}$ have the same initial point.

a b



c

$a+b > c$

$a+c > b$

$b+c > a$

$\vec{v}$ $\vec{w}$

$\vec{v}+\vec{w}$

* From Figure 2.5(a), one can see that

$$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

which is known as the **triangle inequality** (used in homework).

- Vector subtraction
  - Given two vectors $\mathbf{v}, \mathbf{w}$, define their difference $\mathbf{v}-\mathbf{w}$ to be $\mathbf{v}+(-\mathbf{w}) = \mathbf{v}+(-1)\mathbf{w}$.
  - Thus vector subtraction combines vector addition and scalar multiplication.
  - Graphically:

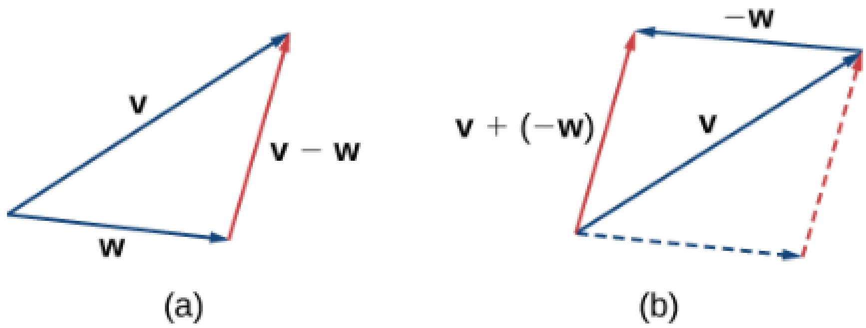
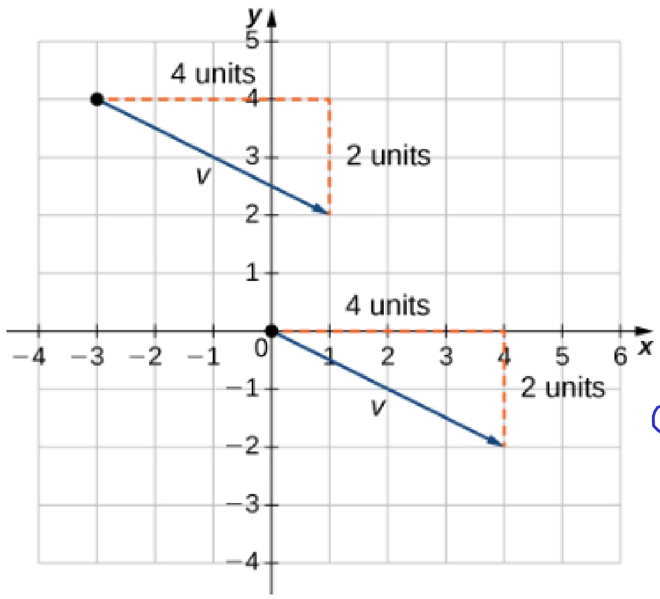


Figure 2.6 (a) The vector difference $\mathbf{v}-\mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$. (b) The vector $\mathbf{v}-\mathbf{w}$ is equivalent to the vector $\mathbf{v}+(-\mathbf{w})$.

- Vector Components
  - It can be easier to work with vectors when we have an underlying coordinate system (Cartesian).
  - Given any vector, we can translate it so that its initial coordinate is at the origin. If that vector has its terminal point at (x, y), then we can write the vector in **component form** as

$$\mathbf{v} = \langle x, y \rangle$$

$$\mathbf{v} = \langle 4, -2 \rangle .$$

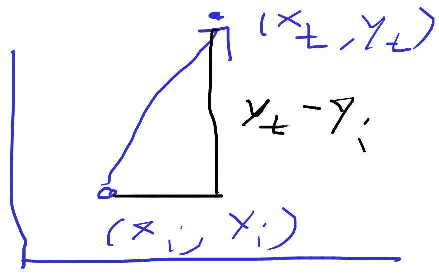


(4,-2)

Figure 2.12 These vectors are equivalent.

  - * Note the difference in bracketing. Parentheses are used to denote a coordinate point. Angle brackets are used to denote a vector.
  - The above process can be completely done in terms of coordinates: A vector $\mathbf{v}$ with initial point (x_i, y_i) and terminal point (x_t, y_t) can be expressed in component form as

$$\mathbf{v} = \langle x_t - x_i, y_t - y_i \rangle$$



> **Example.** Using the previous Figure 2.12, note the original vector $\mathbf{v}$ has initial point $(-3,4)$ and terminal point $(1,2)$. We can compute the component form of $\mathbf{v}$ algebraically:
>
> $$\mathbf{v} = \big\langle 1-(-3), 2-4 \big\rangle = \langle 4, -2 \rangle$$

- The magnitude of a vector $\mathbf{v} = \langle x, y \rangle$ is computed with the formula

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}$$

  - Note that this is just an application of Pythagorean theorem:

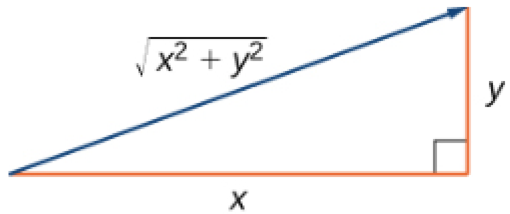


Figure 2.13 If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

- The previous operations can be done in terms of components:
  Let $\mathbf{v} = \langle x_1, y_1 \rangle$, $\mathbf{w} = \langle x_2, y_2 \rangle$. Let k be a scalar.
  Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1 \rangle$
  Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$.

- Vector arithmetic:

Theorem 2.1: Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a plane. Let r and s be scalars.

i.	$\mathbf{u} + \mathbf{v}$	=	$\mathbf{v} + \mathbf{u}$	Commutative property
ii.	$(\mathbf{u} + \mathbf{v}) + \mathbf{w}$	=	$\mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property
iii.	$\mathbf{u} + 0$	=	$\mathbf{u}$	Additive identity property
iv.	$\mathbf{u} + (-\mathbf{u})$	=	0	Additive inverse property
v.	$r(s\mathbf{u})$	=	$(rs)\mathbf{u}$	Associativity of scalar multiplication
vi.	$(r + s)\mathbf{u}$	=	$r\mathbf{u} + s\mathbf{u}$	Distributive property
vii.	$r(\mathbf{u} + \mathbf{v})$	=	$r\mathbf{u} + r\mathbf{v}$	Distributive property
viii.	$1\mathbf{u}$	=	$\mathbf{u}$, $0\mathbf{u} = 0$	Identity and zero properties

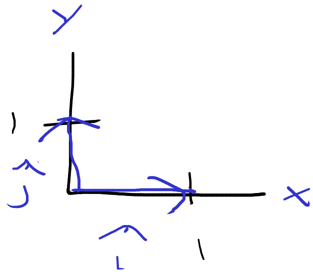
- Unit vectors and normalization
  - A **unit vector** is a vector with magnitude 1.

- For any vector $\mathbf{v}$, we can produce a unit vector in the same direction as $\mathbf{v}$. Simply divide by the length of the vector:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

  We say that $\mathbf{u}$ is the *unit vector in the direction of* $\mathbf{v}$. This process is called **normalization**.

- Unit vectors are often denoted with a hat: $\hat{\mathbf{v}}$.



- Vectors can also be written in terms of the **standard unit vectors**

$$\mathbf{i} = \langle 1, 0 \rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1 \rangle .$$

  One can write

$$\mathbf{v} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}.$$

Similar problems (§2.1 #1, 7, 9, 15, 27)

For the following exercises, consider point $P(-1,3)$, $Q(1,5)$, and $R(-3,7)$. Determine the requested vectors and express each of them a. in component form and b. by using the standard unit vectors.

#1 $\overrightarrow{PQ}$

$\overrightarrow{PQ} = \big\langle 1-(-1), 5-3 \big\rangle = \langle 2, 2 \rangle$ or $2\mathbf{i} + 2\mathbf{j}$.

#7 $2\overrightarrow{PQ} - 2\overrightarrow{PR}$

$\overrightarrow{PR} = \big\langle -3-(-1), 7-3 \big\rangle = \langle -2, 4 \rangle$ so

$$\begin{aligned} 2\overrightarrow{PQ} - 2\overrightarrow{PR} &= 2\langle 2, 2 \rangle + (-2)\langle -2, 4 \rangle \\ &= \langle 4, 4 \rangle + \langle 4, -8 \rangle \\ &= \langle 8, -4 \rangle \;\text{ or } 8\mathbf{i} - 4\mathbf{j} \end{aligned}$$

#9 The unit vector in the direction of $\overrightarrow{PQ}$

Since $\|\overrightarrow{PQ}\| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{1}{2\sqrt{2}}\langle 2, 2 \rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \text{ or } \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

#15 Use the given vectors **a** and **b**:
$\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$.

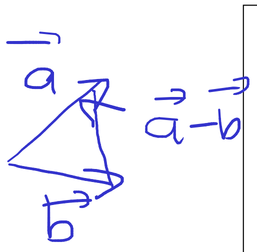
(a) Determine the vector sum $\mathbf{a} + \mathbf{b}$ and express it in both component form and by using the standard unit vectors.

$\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$ or $\langle 3, 4 \rangle$

(b) Find the vector difference and express it in both the component form and by using the standard unit vectors.

$\mathbf{a} - \mathbf{b} = \mathbf{i} - 2\mathbf{j}$ or $\langle 1, -2 \rangle$

(c) Verify that the vectors **a**, **b**, and $\mathbf{a} - \mathbf{b}$ satisfy the triangle inequality.



Considering Figure 2.6(a), see that the vectors do create a triangle. Computing lengths:

$$\|\mathbf{a}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$\|\mathbf{b}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$
$$\|\mathbf{a} - \mathbf{b}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

We see that the triangle inequalities are indeed satisfied (the sum of any two is greater than the third):

$$\sqrt{5} \leq \sqrt{10} + \sqrt{5} \quad \checkmark$$
$$\sqrt{2}\cdot\sqrt{5} = \sqrt{10} \leq 2\sqrt{5} \quad \checkmark$$

$\sqrt{2} \leq 2$

(d) Determine the vectors $2\mathbf{a}$, $-\mathbf{b}$, and $2\mathbf{a} - \mathbf{b}$. Express the vectors in both the component form and by using standard unit vectors.

$$2\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} \text{ or } \langle 4, 2 \rangle$$
$$-\mathbf{b} = -\mathbf{i} - 3\mathbf{j} \text{ or } \langle -1, -3 \rangle$$
$$2\mathbf{a} - \mathbf{b} = 3\mathbf{i} - \mathbf{j} \text{ or } \langle 3, -1 \rangle$$

#27 Find the vector **v** with the given magnitude and in the same direction as vector **u**.

$\|\mathbf{v}\| = 7$, $\mathbf{u} = \langle 3, -5 \rangle$

$$\|\vec{v}\| \frac{\vec{u}}{\|\vec{u}\|}$$

The approach: This vector can be produced by first normalizing $\mathbf{u}$, then scaling it up to have the same magnitude as $\mathbf{v}$. This can be combined into one expression: $\frac{\|\mathbf{v}\|}{\|\mathbf{u}\|}\mathbf{u}$. Computing

$$\|\mathbf{u}\| = \sqrt{3^2 + (-5)^2} = \sqrt{34}$$

The desired vector is thus

$$\frac{7}{\sqrt{34}} \langle 3, -5 \rangle = \left\langle \frac{21}{\sqrt{34}}, -\frac{35}{\sqrt{34}} \right\rangle$$