§2.2: Vectors in Three Dimensions

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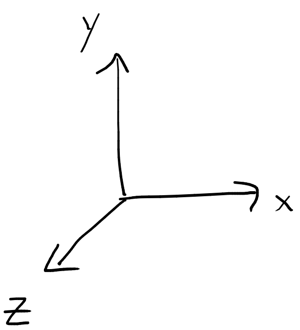
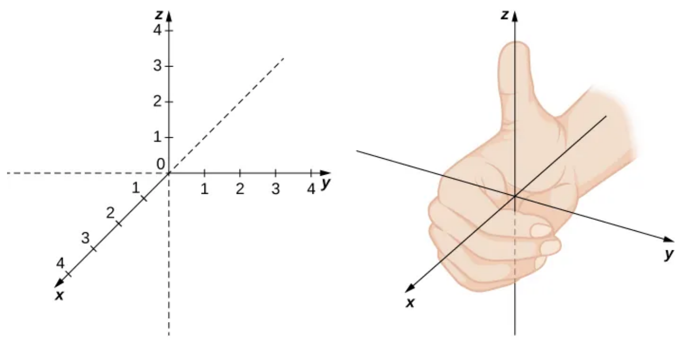
Contents

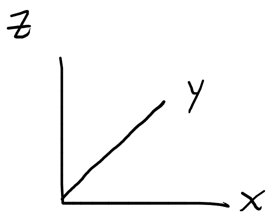
1 Introduction to 3D coordinate systems

The three-dimensional rectangular (Cartesian) coordinate system consists of three perpendicular axes: the x, y, and z-axes. To make the direction of the *positive* z-axis unambiguous, we assert that the **right hand rule** must hold: The positive x, y, z axes should satisfy the right hand rule as shown:



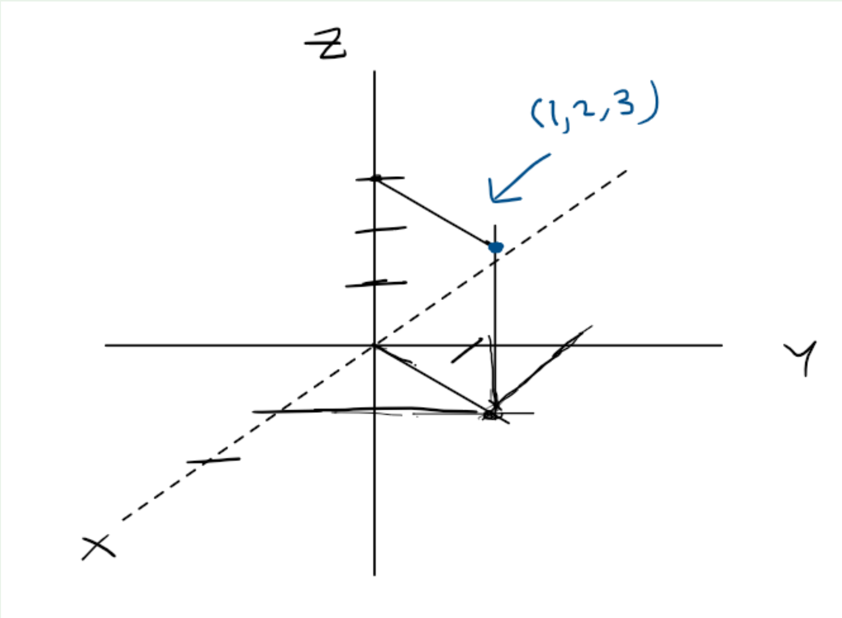


(a) (b)



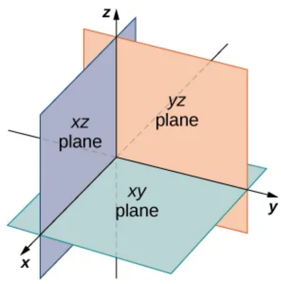
1.1 Sketching things in 3D

Example. Sketching the point $(1,2,3)$



1.2 Coordinate planes

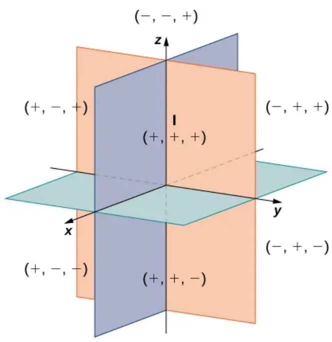
There are three coordinate planes.



- xy-plane: $\{(x, y, 0) : x, y \in \mathbb{R}\}$
- xz-plane: $\{(x, 0, z) : x, z \in \mathbb{R}\}$
- yz-plane: $\{(0, y, z) : y, z \in \mathbb{R}\}$

1.3 Octants

Similar to how in 2D, the coordinate axes partition the plane into four quadrants, in 3D, the coordinate planes divide space into eight **octants**.

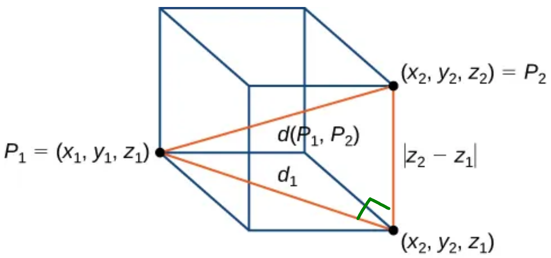


1.4 3D distance formula

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This formula can be arrived at from the 2D distance formula by considering the following diagram:



$$\begin{aligned} d_1 &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ \implies d &= \sqrt{d_1^2 + (z_2 - z_1)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}$$

Example (Similar to Exercise 2.62)**.** Determine the distance between the point $P(3, 2, 6)$ and the origin.

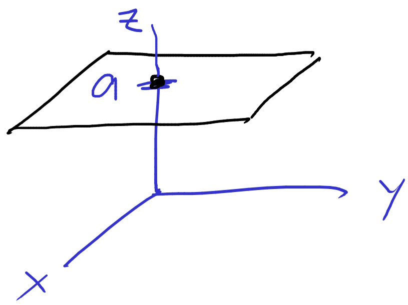
Solution. $\sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

2 Equations in $\mathbb{R}^3$

The coordinate planes can be written as equations:

$z = a$

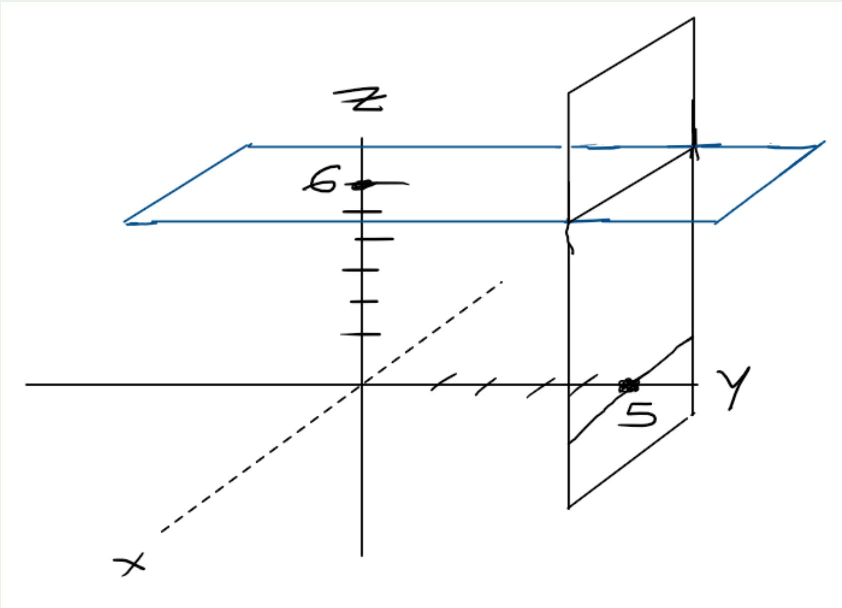
- xy-plane: $z = 0$
- xz-plane: $y = 0$
- yz-plane: $x = 0$



Changing the value '0' translates these planes.

Example (Similar to Exercise 2.64)**.** Describe and graph the set of points that satisfies the equation $(y-5)(z-6)=0$

Answer. This is the union of two planes, $y=5$ and $z=6$. Graph:



2.1 Equation of a sphere

In 2D, the equation of a circle of radius r centered at (h,k) is

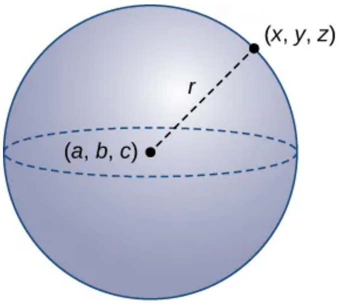
$$(x-h)^2+(y-k)^2=r^2$$

Generalizing this to 3D, the **equation of a sphere** of radius r centered at (a,b,c) is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

This formula can be derived by considering the distance formula and the definition of a sphere.

Definition. A **sphere** is the set of all points in space equidistant from a fixed point, the center of the sphere. This distance from the center to a point on the sphere is called the **radius**.



$$d = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$$

Example (Similar to Exercise 2.72)**.** Find the equation of the sphere in standard form with center $C(-1, 7, 4)$ and radius 4.

Answer. $(x+1)^2 + (y-7)^2 + (z-4)^2 = 16$

3 Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are of the form $\langle x, y, z\rangle$. The zero vector is $\mathbf{0} = \langle 0, 0, 0\rangle$.

Vector addition and scalar multiplication If $\mathbf{v} = \langle x_1, y_1, z_1\rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2\rangle$ are vectors and, k is a scalar, then

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle \\ k\mathbf{v} &= \langle kx_1, ky_1, kz_1\rangle . \end{aligned}$$

Magnitude If $\mathbf{v} = \langle x, y, z\rangle$,

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

Normalization If $\mathbf{v} = \langle x, y, z\rangle$,

$$\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle x, y, z\rangle$$

Standard unit vectors in 3D

$$\begin{aligned} \mathbf{i} &= \langle 1, 0, 0\rangle & &x\text{-direction} \\ \mathbf{j} &= \langle 0, 1, 0\rangle & &y\text{-direction} \\ \mathbf{k} &= \langle 0, 0, 1\rangle & &z\text{-direction} \end{aligned}$$

Vectors can be written in component form or using standard unit vectors:

$$\mathbf{v} = \langle x, y, z\rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Example (Similar to Exercise 2.82)**.** Find the terminal point Q of the vector $\overrightarrow{PQ} = \langle 7, -1, 3 \rangle$ with the initial point at $P(-2, 3, 5)$.

Solution. Let $Q = (x, y, z)$. We know $\overrightarrow{PQ} = \langle 7, -1, 3 \rangle = \left\langle x - (-2), y - 3, z - 5 \right\rangle$, or

$$\begin{cases} 7 = x + 2 \\ -1 = y - 3 \\ 3 = z - 5 \end{cases}$$

Solving, we get $x = 5, y = 2, z = 8$, so $Q = (5, 2, 8)$

Example (Similar to Exercise 2.92)**.** Find the unit vector in the direction of the given vector $\mathbf{a}$ and express it using standard unit vectors.
$\mathbf{a} = \langle -2, 4, 5 \rangle$

Solution.

$$\|\mathbf{a}\| = \sqrt{(-2)^2 + 4^2 + 5^2} = \sqrt{45} = 3\sqrt{5}$$
$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left\langle \frac{-2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}} \right\rangle = \frac{-2}{3\sqrt{5}}\mathbf{i} + \frac{4}{3\sqrt{5}}\mathbf{j} + \frac{5}{3\sqrt{5}}\mathbf{k}$$