§2.3: The Dot Product

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1 Brief Discussion of Products for Vectors

In this course, we will encounter three different kinds of vector products: the dot product, cross product, and triple scalar product.

These operations are quite from one another, which can be seem from their "signatures":

$$
\begin{aligned}
- \cdot - &: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\
- \times - &: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \\
- \cdot (- \times -) &: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}
\end{aligned}
$$

The key distinction here:

- The dot product takes *two* vectors (in n-space) and returns a *scalar*
- The cross product takes *two* vectors (in 3-space) and returns a *vector*
- The triple scalar product takes *three* vectors (in 3-space) and returns a *scalar*

2 The Dot Product and Its Properties

Definition. The **dot product** of two vectors $\mathbf{u} = \langle u_1, \ldots u_n \rangle$, $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

In particular for this course, we will work with $n = 2, 3$, where the dot product looks like the following:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad (n = 3)$$
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 \qquad (n = 2)$$

Example (Similar to Exercise 2.126)**.** Some example computations of dot product:

$$\langle 4, 5 \rangle \cdot \langle 1, 2 \rangle = 4 + 10 = 14$$
$$\langle 1, 4, -2 \rangle \cdot \langle 2, -3, 1 \rangle = 2 + (-12) + -2 = -12$$

Several algebraic properties of the dot product:

Theorem 2.3: Properties of the Dot Product

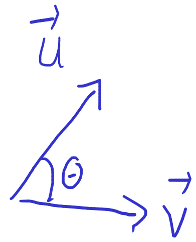
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors, and let c be a scalar.

i.	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	Commutative property
ii.	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	Distributive property
iii.	$c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$	Associative property
iv.	$\mathbf{v} \cdot \mathbf{v} = \| \mathbf{v} \|^2$	Property of magnitude

Proof. We only prove part (iv) for dimension 3. Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + v_3^2$$
$$\|\mathbf{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2} \right)^2$$

which are clearly equal. □



3 Dot Product-Angle Formula

There is a relationship between the dot product of two vectors and the angle between them:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \qquad \text{or} \qquad \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

For the derivation of this, see the textbook: OpenStax Calc 3, Theorem 2.4.

Remark. Since the angle between two vectors is typically described within the range $[0, \pi]$, which agrees with the range of the arccosine function, it is safe to take the inverse cosine of both sides to get

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right)$$

This is in anticipation of a similar formula we will see relating the cross product with sine, where an issue will arise.

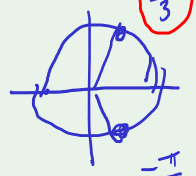
Example (Similar to Exercise 2.138)**.** Find the measure of the angle between the three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$. Express the answer in radians rounded to two decimal places, if it is not possible to express is exactly.
$\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} - \mathbf{k}$

Solution.

$\cos ? = \frac{1}{2}$

$\frac{\pi}{3}$

$$\mathbf{a} \cdot \mathbf{b} = 1(0) + 1(1) + 0(-1) = 1$$
$$\|\mathbf{a}\| = \sqrt{1+1} = \sqrt{2}$$
$$\|\mathbf{b}\| = \sqrt{1+(-1)^2} = \sqrt{2}$$
$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}\right)$$
$$= \arccos\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$$



3.1 Orthogonality

Two vectors $\mathbf{v}, \mathbf{w}$ are **orthogonal** or **perpendicular** if the angle between them is $\frac{\pi}{2}$. We may write $\mathbf{v} \perp \mathbf{w}$ to express this. The zero vector is considered to be orthogonal to all other vectors.

The dot product can be used to test whether $\mathbf{v}$ and $\mathbf{w}$ are orthogonal:

$$\boxed{\mathbf{v} \perp \mathbf{w} \quad \text{if and only if} \quad \mathbf{v} \cdot \mathbf{w} = 0}$$

Proof. Assume $\mathbf{v}, \mathbf{w}$ are not zero, as otherwise the statement clearly holds. Then

$$\mathbf{v} \cdot \mathbf{w} = 0 \iff \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$$

□

Example. The standard basis vectors are mutually orthogonal and have length 1. We can quickly see

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0 \qquad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$v \cdot v = \|v\|^2$

Example. Determine whether $\mathbf{v} = \langle 2, 6, 1 \rangle$ is orthogonal to $\mathbf{u} = \langle 2, -1, 1 \rangle$ or $\mathbf{w} = \langle -4, 1, 2 \rangle$.

Solution.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{u} &= \langle 2, 6, 1 \rangle \cdot \langle 2, -1, 1 \rangle = 2(2) + 6(-1) + 1(1) = -1 && \text{(not orthogonal)} \\ \mathbf{v} \cdot \mathbf{w} &= \langle 2, 6, 1 \rangle \cdot \langle -4, 1, 2 \rangle = 2(-4) + 6(1) + 1(2) = 0 && \text{(orthogonal)} \end{aligned}$$

Example (Similar to Exercise 2.142)**.** Determine whether the given vectors are orthogonal:
$\mathbf{a} = \langle x, y \rangle$, $\mathbf{b} = \langle -y, x \rangle$, where x and y are nonzero real numbers.

Solution. $\mathbf{a} \cdot \mathbf{b} = x(-y) + yx = 0$ so these two vectors $\boxed{\text{are orthogonal}}$

3.2 Using dot product to classify angles

Considering the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right)$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

further, we can arrive at the following classification:

$$\begin{array}{ll} \mathbf{u} \cdot \mathbf{v} > 0 & \theta \text{ acute angle} \\ \mathbf{u} \cdot \mathbf{v} = 0 & \theta \text{ right angle} \\ \mathbf{u} \cdot \mathbf{v} < 0 & \theta \text{ obtuse angle} \end{array}$$

cos

$\vec{u} \cdot \vec{v} > 0 \iff \cos\theta > 0 \iff \theta$ in I or IV $\iff \theta$ acute

4 Vector Projection

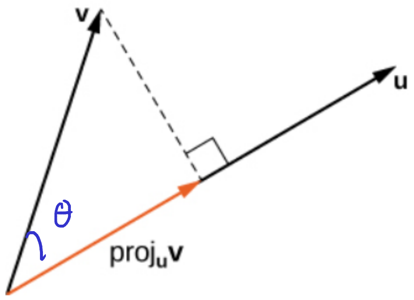


Figure 2.50 The projection of **v** onto **u** shows the component of vector **v** in the direction of **u**.

Definition. The **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$ is denoted by $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ and can be computed with any of the following expressions:

$$\begin{aligned}\text{proj}_{\mathbf{u}}\,\mathbf{v} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right)\mathbf{u}\\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\right)\frac{\mathbf{u}}{\|\mathbf{u}\|}\end{aligned}$$

The length of this vector is also known as the **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$ and is denoted by

$$\|\text{proj}_{\mathbf{u}}\,\mathbf{v}\| = \text{comp}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}$$

Derivation of this formula Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$. From the diagram, we see

$$\begin{aligned}\cos\theta &= \frac{\|\text{proj}_{\mathbf{u}}\,\mathbf{v}\|}{\|\mathbf{v}\|}\\ \implies \|\text{proj}_{\mathbf{u}}\,\mathbf{v}\| &= \|\mathbf{v}\|\cos\theta\\ &= \|\mathbf{v}\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\end{aligned}$$

Example (Similar to Exercise 2.168)**.** Let $\mathbf{u} = 3\mathbf{i} + 2\mathbf{k}, \mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$.

(a) Find the vector projection $\mathbf{w} = \operatorname{proj}_{\mathbf{u}} \mathbf{v}$. Express in component form.

Answer.

$$\begin{aligned}\operatorname{proj}_{\mathbf{u}} \mathbf{v} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \\ &= \frac{0+0+8}{9+0+4}\langle 3,0,2\rangle \\ &= \frac{8}{13}\langle 3,0,2\rangle \\ &= \boxed{\left\langle \frac{24}{13}, 0, \frac{16}{13}\right\rangle}\end{aligned}$$

(b) Find the scalar projection $\operatorname{comp}_{\mathbf{u}} \mathbf{v}$.

Answer.

$$\begin{aligned}\operatorname{comp}_{\mathbf{u}} \mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|} \\ &= \boxed{\frac{8}{\sqrt{13}}}\end{aligned}$$

5 Work

In physics, **work** is the energy transferred to or from an object via the application of force along a displacement.

You may have seen in a previous class the formula $W = Fd$ (work equals force times displacement). This is true if the force is constant and is being applied in the same direction as the travel of the object.

If instead, the constant force $\mathbf{F}$ is being applied at an angle θ from the displacement of the object (the displacement denoted by $\mathbf{s}$), then we use the formula

$$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\|\,\|\mathbf{s}\| \cos\theta$$

Example (similar to Exercise 2.177)**.** A child is pulling a wagon with a force of 8lb at an angle of 55°. The child pulls the wagon 50 ft. Find the work done by the force.

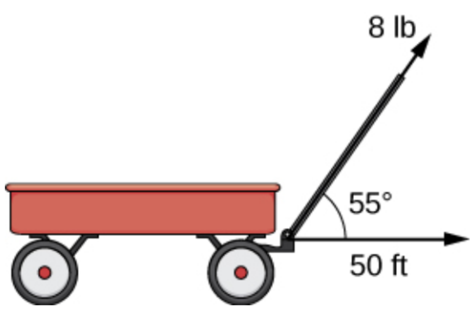


Figure 2.51 The horizontal component of the force is the projection of **F** onto the positive *x*-axis.

Solution.

$$\begin{aligned} W &= \|\mathbf{F}\|\,\|\mathbf{s}\| \cos\theta \\ &= 8(50)\cos(55^\circ) \approx 229.431 \text{ ft-lbs} \end{aligned}$$