Contents

1 Determinants, briefly

In order to define the cross product algebraically, it is convenient to use the notion of a determinant. The **determinant** is a function/operation that takes a *square* matrix and returns a number.

- 2×2 determinant formula:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

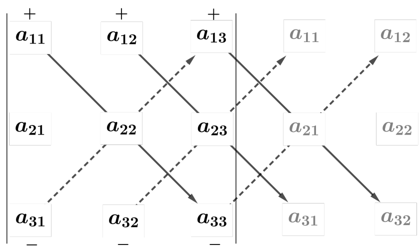
  - "main diagonal minus anti-diagonal"

- 3×3 determinant formula:

$+ \; - \; +$
$- \; + \; -$
$+ \; - \; +$

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei + bfg + cdh - afh - bdi - ceg \end{aligned}$$

  - The first equality is cofactor expansion using the top row.
    - Cofactor expansion can be used done using any row or column. The signs in front alternate (like a checkerboard).
  - The last equality has a mnemonic: positive diagonals minus negative diagonals, with wrapping
    - This mnenonic fails for matrices larger than 3×3.



2 Cross product

The cross product is an operation which takes two vectors in $\mathbb{R}^3$ and returns a vector in $\mathbb{R}^3$.

2.1 Algebraic definition

Definition. The cross product of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\begin{aligned} \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle \end{aligned}$$

Example (Similar to Exercise 2.190, 2.198)**.** Let $\mathbf{u} = \langle 3, -1, 2 \rangle$, $\mathbf{v} = \langle -2, 0, 1 \rangle$. Find the unit vector $\mathbf{w}$ in the direction of the cross product vector $\mathbf{u} \times \mathbf{v}$. Express your answer using standard unit vectors.

Solution.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -2 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ -2 & 0 \end{vmatrix} \mathbf{k} \\ &= (-1 - 0)\mathbf{i} - (3 - (-4))\mathbf{j} + (0 - 2)\mathbf{k} \\ &= -\mathbf{i} - 7\mathbf{j} - 2\mathbf{k} \\ \|\mathbf{u} \times \mathbf{v}\| &= \sqrt{1 + 49 + 4} = \sqrt{54} = 3\sqrt{6} \\ \mathbf{w} = \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} &= \boxed{-\frac{1}{3\sqrt{6}}\mathbf{i} - \frac{7}{3\sqrt{6}}\mathbf{j} - \frac{2}{3\sqrt{6}}\mathbf{k}} \end{aligned}$$

2.2 Geometric definition of cross product

Given two nonzero nonparallel vectors $\mathbf{u}, \mathbf{v}$, the cross product $\mathbf{u} \times \mathbf{v}$ is the unique vector with the following three properties:

(i) $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$

(ii) $\mathbf{u} \times \mathbf{v}$ has length $\|\mathbf{u}\| \|\mathbf{v}\| \sin\theta$. This is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

$\sin\theta = \frac{x}{\|v\|}$

$\Rightarrow \|v\| \sin\theta = x$

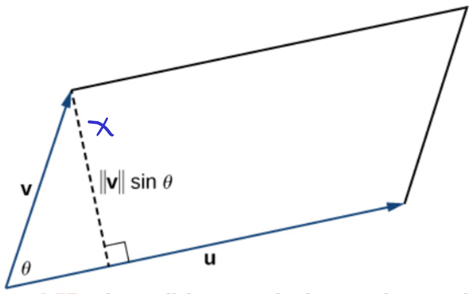


Figure 2.57 The parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ has base $\|\mathbf{u}\|$ and height $\|\mathbf{v}\| \sin\theta$.

$$\begin{aligned} \text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\mathbf{u}\|(\|\mathbf{v}\| \sin\theta) \end{aligned}$$

(iii) $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ forms a right-handed system.

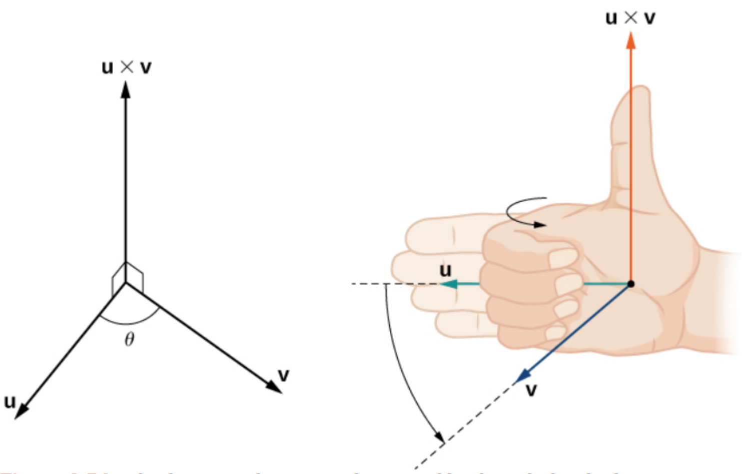


Figure 2.54 The direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule.

Example (Similar to Exercise 2.209)**.** Find the area of the paralleogram with adjacent sides $\mathbf{u} = \langle 1, 2, -3\rangle$ and $\mathbf{v} = \langle 0, 2, 1\rangle$.

Solution. The area of such a parallelogram given by $\|\mathbf{u} \times \mathbf{v}\|$. So we compute

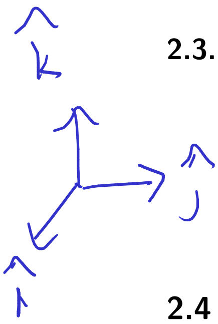
$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -3 \\ 0 & 2 & 1 \end{vmatrix} \\ &= (2+6)\mathbf{i} - (1-0)\mathbf{j} + (2-0)\mathbf{k} \\ &= \langle 8, -1, 2\rangle \\ \|\mathbf{u} \times \mathbf{v}\| &= \sqrt{64+1+4} = \boxed{\sqrt{69}} \end{aligned}$$

2.3 Algebraic properties of the cross product

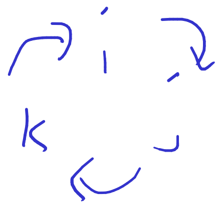
Theorem 2.6: Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

i.	$\mathbf{u} \times \mathbf{v}$	$=$	$-(\mathbf{v} \times \mathbf{u})$	Anticommutative property
ii.	$\mathbf{u} \times (\mathbf{v} + \mathbf{w})$	$=$	$\mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$	Distributive property
iii.	$c(\mathbf{u} \times \mathbf{v})$	$=$	$(c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$	Multiplication by a constant
iv.	$\mathbf{u} \times \mathbf{0}$	$=$	$\mathbf{0} \times \mathbf{u} = \mathbf{0}$	Cross product of the zero vector
v.	$\mathbf{v} \times \mathbf{v}$	$=$	$\mathbf{0}$	Cross product of a vector with itself
vi.	$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$	$=$	$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$	Scalar triple product



2.3.1 Cross product on the standard unit vectors



$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

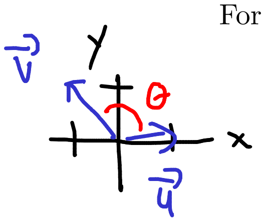
$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

2.4 Cross product–angle formula

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta \quad \text{or} \quad \sin\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Remark. It is not true in general, that

$$\theta = \sin^{-1} \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

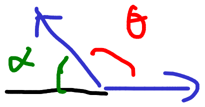


For instance, consider the vectors $\mathbf{u} = \langle 1, 0, 0\rangle$, $\mathbf{v} = \langle -1, 1, 0\rangle$. Computing gives

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \mathbf{k} = 1\mathbf{k}$$

$$\sin\theta = \frac{\|\langle 0, 0, 1\rangle\|}{\sqrt{2} \cdot 1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

which is true, but plugging $\sin^{-1}(\frac{\sqrt{2}}{2})$ into a calculator would give $\frac{\pi}{4}$, instead of the correct answer, $\frac{3\pi}{4}$. Instead, the correct statement is



$$\theta = \begin{cases} \sin^{-1} \frac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} & \text{if } \theta \text{ acute} \\ \pi - \sin^{-1} \frac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} & \text{if } \theta \text{ obtuse} \end{cases}$$

Moral of this story: If you want to find the angle between two vectors, use the dot product–angle formula instead.

3 Triple scalar product

The triple scalar product of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $\mathbf{w} = \langle w_1, w_2, w_3\rangle$,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

3.1 Algebraic properties

- Cyclic shift remains equal

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

- A single swap negates:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \\ &= -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) \\ &= -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) \end{aligned}$$

3.2 Geometric property: Volume of parallelopiped

The parallelopiped spanned by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ has volume

$$V = \left|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\right|$$

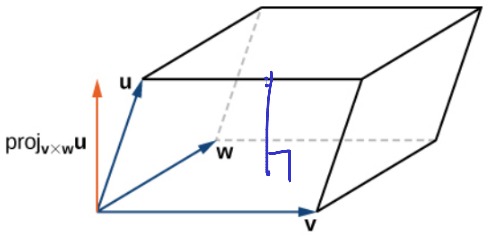


Figure 2.59 The height of the parallelepiped is given by $\| \mathrm{proj}_{\mathbf{v}\times\mathbf{w}} \mathbf{u} \|$.

Derivation:

$$\begin{aligned} V &= \left\|\mathrm{proj}_{\mathbf{v}\times\mathbf{w}}\, \mathbf{u}\right\| \|\mathbf{v} \times \mathbf{w}\| \\ &= \left|\frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|}\right| \|\mathbf{v} \times \mathbf{w}\| \\ &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \end{aligned}$$

Example (Similar to Exercise 2.213)**.** Let $\mathbf{u} = \langle -3, 5, -1\rangle$, $\mathbf{v} = \langle 0, 2, -2\rangle$, $\mathbf{w} = \langle 3, 1, 1\rangle$.

(a) Find the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Solution.

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} -3 & 5 & -1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 2\begin{vmatrix} -3 & -1 \\ 3 & 1 \end{vmatrix} - (-2)\begin{vmatrix} -3 & 5 \\ 3 & 1 \end{vmatrix} \\ &= 2(-3 - (-3)) + 2(-3 - 15) = \boxed{-36} \end{aligned}$$

(b) Find the volume of the parallelopiped with the adjacent edges $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Solution. The volume of such a paralleopiped is given by $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. So

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \boxed{36}$$