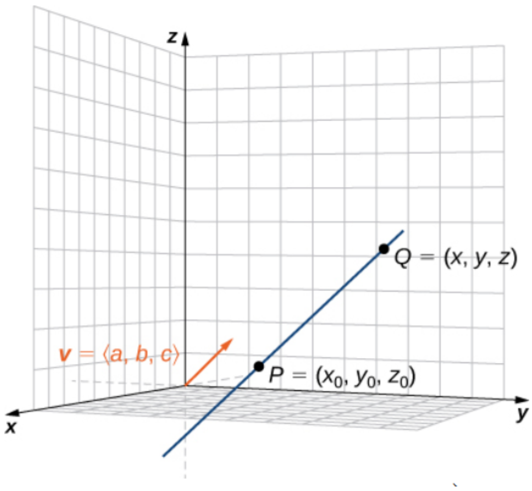
Contents

1 Descriptions of a line



Let the line L contain two points, $P(x_0, y_0, z_0)$ which is fixed, and $Q(x, y, z)$ which varies along L. Let $\mathbf{v} = \langle a, b, c\rangle$ capture the direction that the line L extends, which we call the **direction vector**. It has the same direction as $\overrightarrow{PQ}$.

Vector equation of a line

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle \qquad (t \in \mathbb{R})$$

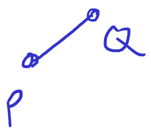
Parametric equations of a line

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \qquad (t \in \mathbb{R})$$

Symmetric equations of a line

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}.$$

Remark. Isolating at the components of the vector form gives the parametric form. Solving each of those equations for t and setting them all equal gives the symmetric form.



1.1 Descriptions of a line segment

Let $P(x_0, y_0, z_0)$, $Q(x_1, y_1, z_1)$ be the endpoints of a line segment. Let $\mathbf{p} = \langle x_0, y_0, z_0\rangle$ and $\mathbf{q} = \langle x_1, y_1, z_1\rangle$ be their associated position vectors.

Vector equation of a line segment

$\langle x, y, z\rangle = (1-t)\langle x_0, y_0, z_0\rangle + t\langle x_1, y_1, z_1\rangle$

$$\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q}, \quad (0 \le t \le 1)$$

Parametric equations for a line segment

$x = (1-t)x_0 + t x_1$
$= x_0 + t(x_1 - x_0)$

$$\begin{cases} x = x_0 + t\,(x_1 - x_0) \\ y = y_0 + t\,(y_1 - y_0) \\ z = z_0 + t\,(z_1 - z_0) \end{cases} \qquad (0 \le t \le 1)$$

Example (Similar to Exercise 2.244)**.** Let L be the line passing through points $P(-3, 5, 9)$ and $Q(4, -7, 2)$.

$\vec{v} = \langle 4, -7, 2\rangle - \langle -3, 5, 9\rangle$
$= \langle 7, -12, -7\rangle$

(a) Find the vector equation of line L.

Answer. $\mathbf{r} = \langle -3, 5, 9\rangle + t\,\langle 7, -12, -7\rangle, \quad t \in \mathbb{R}$

(b) Find parametric equations of line L.

Answer. $x = -3 + 7t,\ \ y = 5 - 12t,\ \ z = 9 - 7t, \quad t \in \mathbb{R}$

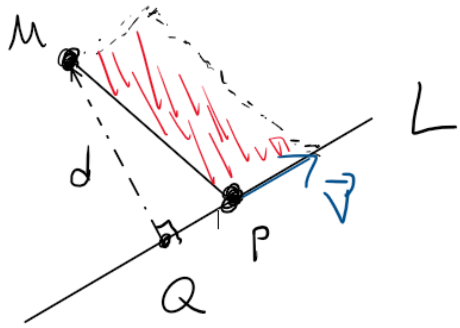
(c) Find symmetric equations of line L.

Answer. $\dfrac{x+3}{7} = \dfrac{y-5}{-12} = \dfrac{z-9}{-7}$

(d) Find parametric equations of the line segment determined by P and Q.

Answer. $x = -3 + 7t,\ \ y = 5 - 12t,\ \ z = 9 - 7t, \quad t \in [0, 1]$

2 Distance between a point and a line



Proposition. Let L be a line with direction vector $\mathbf{v}$ and let M be a point not on L. The distance from M to L can be computed by the formula

$$d = \frac{\left\| \overrightarrow{PM} \times \mathbf{v} \right\|}{\|\mathbf{v}\|}$$

where P is any point on the line L.

Proof. The area of the parallelogram (in red) can be expressed in two ways, which must be equal:

$$\text{Area} = \|\overrightarrow{PM} \times \mathbf{v}\| = \|\mathbf{v}\| d$$

□

Example (Similar to Exercise 2.253)**.** Find the distance between point $A(2, 1, -3)$, and the line of symmetric equations $x = 2y = -z$.

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

Solution. The point $P = (0, 0, 0)$ is clearly on the line. The line defined by the symmetric equations has direction vector $\mathbf{v} = \langle 2, 1, -2 \rangle$. A is not on the line. Thus

$$d = \frac{\left\| \overrightarrow{PA} \times \mathbf{v} \right\|}{\|\mathbf{v}\|} = \frac{\| \langle 2, 1, -3 \rangle \times \langle 2, 1, -2 \rangle \|}{3} = \frac{\| \langle 1, -2, 0 \rangle \|}{3} = \boxed{\frac{\sqrt{5}}{3}}$$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 2 & 1 & -2 \end{vmatrix} = \langle -2+3, -(-4+6), 0 \rangle = \langle 1, -2, 0 \rangle$$

3 Descriptions of a plane

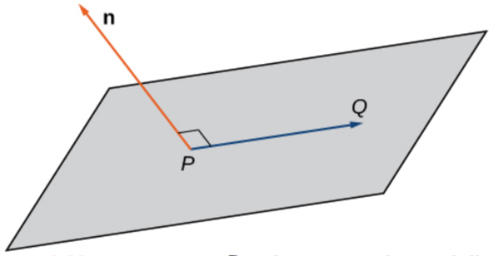


Figure 2.69 Given a point P and vector $\mathbf{n}$, the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\mathbf{n}$ forms a plane.

A plane with normal vector $\mathbf{n} = \langle a, b, c\rangle$ and passing through a point $P = (x_0, y_0, z_0)$ can be expressed in the following ways:

$Q = (x, y, z)$

Vector equation of a plane:

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$

$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

Scalar equation of a plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

General form of the equation of a plane

$$ax + by + cz + d = 0$$

where $d = -ax_0 - by_0 - cz_0$.

Remark. Expanding the dot product in the vector equation form gives the scalar equation form. Distributing that gives the general form.

Example (Similar to Exercise 2.268). Let $P = (1, 2, 3)$ and $\mathbf{n} = \langle 4, 5, 6\rangle$.

(a) Find the scalar equation of the plane that passes through P and has normal vector $\mathbf{n}$.

Answer. $4(x - 1) + 5(y - 2) + 6(z - 3) = 0$

(b) Find the general form of the equation of the plane that passes through P and has normal vector $\mathbf{n}$.

Answer. $4x + 5y + 6z - 32 = 0$

4 Distance between a plane and a point

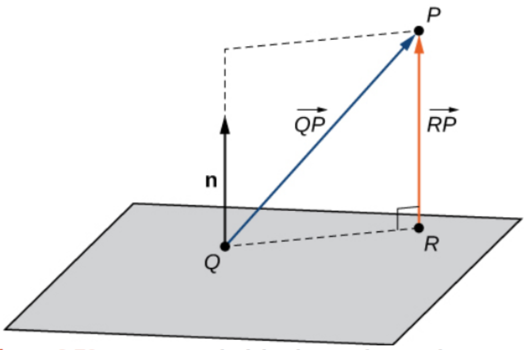


Figure 2.70 We want to find the shortest distance from point P to the plane. Let point R be the point in the plane such that, for any other point in the plane Q, $\| \overrightarrow{RP} \| < \| \overrightarrow{QP} \|$.

Suppose a plane with normal vector $\mathbf{n}$ passes through a point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \left\| \operatorname{proj}_{\mathbf{n}} \overrightarrow{QP} \right\| = \left| \operatorname{comp}_{\mathbf{n}} \overrightarrow{QP} \right| = \frac{\left| \overrightarrow{QP} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}$$

Theorem (2.14: Distance from a Point to a Plane)**.** Let $P(x_0, y_0, z_0)$ be a point. The distance from P to the plane $ax + by + cz + k = 0$ is given by

$ax_1 + by_1 + cz_1 + k = 0$

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof. The plane has normal vector $\mathbf{n} = \langle a, b, c \rangle$. Taking $Q(x_1, y_1, z_1)$ to be some point on the plane we get

$$\begin{aligned} d = \frac{\left| \overrightarrow{QP} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|} &= \frac{| \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle \cdot \langle a, b, c \rangle |}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

□

Example (Similar to Exercise 2.289)**.** Find the distance from point $P(1, 2, 3)$ to the plane of the equation $4x + 5y - 6z + 7 = 0$.

Solution. $d = \dfrac{|4(1) + 5(2) - 6(3) + 7|}{\sqrt{4^2 + 5^2 + (-6)^2}} = \boxed{\dfrac{3}{\sqrt{77}}}$